L-functions

Last week:

- def$^{\underline{n}}$ of Galois repns
- "fundamental theorem of infinite Galois extensions"

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \varprojlim_{M} \mathrm{Gal}(M/\mathbb{Q})$$

  - Krull top.
  - {closed, normal} $\leftrightarrow$ {Galois ext}
  - {open} $\longleftrightarrow$ {finite ext}
  - {open normal} $\leftrightarrow$ {finite Galois}

- Cyclotomic characters, Ramification, and complex repns

Like last week, this talk is based on Sam Marks's lecture notes.

Artin L-functions

first, an example of an L-function.

A Dirichlet L-function for a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ is

$$L(\chi, s) = \sum_{n \geq 1} \chi(n) n^{-s}$$

Converges in half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

This has the "<u>Euler product</u>"

$$L(\chi, s) = \prod_{p \nmid n} \frac{1}{1 - \chi(p) p^{-s}}$$

We'll define L-functions using the Euler product.

Def$\underline{^n}$. Let L/K be a Galois extension of number fields, and let

$$\rho : \mathrm{Gal}(L/K) \longrightarrow GL(V)$$

V a n-dimensional complex space.

Let $\mathfrak{p}$ be a prime in K lying under a prime $\mathfrak{P}$ in L.

~ If $I_{\mathfrak{P}}$ is a choice of an inertia group at $\mathfrak{p}$, let

$$V^{I_{\mathfrak{P}}} = \{ v \in V \mid \rho(\sigma)v = v \ \forall \sigma \in I_{\mathfrak{P}} \}.$$

We then have a representation

$$\rho^{I_p} : \mathrm{Gal}(L/K) \to GL(V^{I_{\mathfrak{P}}})$$

unramified at $\mathfrak{p}$ coming from ρ.

We now can define functions, one for each prime $\mathfrak{P}$ of K, called local L-factors

$$L_{\mathfrak{P}}(\rho, T) = \frac{1}{\det(1 - T\sigma_{\mathfrak{p}} \mid \rho^{I_{\mathfrak{P}}})} \in \mathbb{C}(T).$$

The local L-factor does not depend on the choice of $\mathfrak{P}$.

With these local L-factors, we can define the global L-function

$$L(\rho, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\rho, N\mathfrak{p}^{-s})$$

where $N\mathfrak{p} = \#\mathbb{F}_{\mathfrak{p}}$ is the norm of $\mathfrak{p}$ and $s \in \mathbb{C}$ such that the product converges.

Let's consider an example.

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_1(\mathbb{C}).$$

ρ is unramified at every prime. Our local L-factors are

$$L_p(\rho, T) = \frac{1}{1-T},$$

and we get the L-function

$$L(\rho, s) = \prod_p \frac{1}{1-p^{-s}},$$

which is the Euler product for

$$\zeta(s) = \sum_{n \geq 1} n^{-s},$$

the Riemann zeta function.

We can generalize this to any number field K.

We get the L-function for

$$\rho : Gal(\bar{K}/K) \longrightarrow GL_1(\mathbb{C})$$

defined by

$$L(\rho, s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}.$$

This is the Dedekind zeta function

$$\zeta_K(s) = \sum_I NI^{-s} \quad \text{of } K,$$

I: nonzero ideals of $\mathcal{O}_K$,
N: absolute norm of I.

Ex. Dirichlet characters show up a lot in analytic number theory.

We can view a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ as a Galois repn' ρ_χ

$$G_{\mathbb{Q}} \twoheadrightarrow Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\chi} GL_1(\mathbb{C})$$

If χ is a primitive character,

$$L(\chi, s) = L(\rho_\chi, s).$$

See Mark's's lecture notes.

I gave a promise: Artin L-functions are well-behaved.

Theorem

Let $\rho: \mathrm{Gal}(L/K) \longrightarrow GL_n(\mathbb{C})$ be a Galois representation. Then there is a function $L_\infty(\rho, s)$ and an integer $c(\rho)$ such that the completed L-function

$$\Lambda(\rho, s) = c(\rho)^{s/2} L(\rho, s) L_\infty(\rho, s)$$

has a meromorphic continuation to $\mathbb{C}$, and

$$\Lambda(\rho, s) = W(\rho) \Lambda(\bar{\rho}, 1-s),$$

where $\bar{\rho}$ is the complex conjugate of ρ and $W(\rho) \in S^1 \subseteq \mathbb{C}^\times$ is a complex number of modulus 1.

(tad)) Pf sketch

~ Reduce to $n = 1$, obtain Dirichlet L-function

~ associate with Hecke character (automorphic)
$\longrightarrow$ lots of tools

$\square$

What is L_∞? Why is Λ "completed"?

Primes $\mathfrak{p}$ of K have associated nonarchimedean absolute values $|\cdot|_{\mathfrak{p}}$ on K.

We get archimedean absolute values induced on K by embedding

$$K \hookrightarrow \mathbb{C}.$$

and to these archimedean absolute values we associate "infinity primes" and $L_\infty(\rho, s)$ is the product of infinity prime local L-factors.

These factors "complete" Λ.

Now for the fun stuff:

Composite systems of ℓ-adic Representations

Def$^{\underline{n}}$ Fix $n \geq 1$. For each prime ℓ, let

$$\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{Q}_\ell)$$

be an ℓ-adic representation.

We say that $\{\rho_\ell\}_\ell$ is a <u>compatible system of ℓ-adic representations</u> if

1) there is a finite set S of primes so that each ρ_ℓ is unramified at all $p \notin S \cup \{\ell\}$

2) for each prime p, $\det(1 - T\sigma_p | \rho_\ell^{I_p}) \in \mathbb{Q}_\ell[T]$ lives in $\mathbb{Q}[T]$ and doesn't depend on ℓ.

If $\rho = \{\rho_\ell\}_\ell$ is a compatible system of ℓ-adic representations, we have local L-factors

$$L_p(\rho, T) = \frac{1}{\det(1 - T\sigma_p \mid \rho_\ell^{I_p})}$$

independent of $\ell \neq p$, and we have L-function

$$L(\rho, s) = \prod_p L_p(\rho, p^{-s})$$

Ex. Tate modules of an elliptic curve give us a compatible system of ℓ-adic repns

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Meromorphic continuation?

functional equations?

NOT SURE

Best bet: associate L-functions to automorphic objects. This worked for elliptic curves.

(Taniyama modularity Thm)

—Wiles 1995

—CDTB 2001

Compatible systems are one type of family of ℓ-adic repns. What about other families?

What about

$$\rho : G_{\mathbb{Q}_p} \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})?$$

$$\rho : G_{\mathbb{Q}_p} \longrightarrow GL_n(\overline{\mathbb{Q}_p})?$$

$\rightarrow$ p-adic Hodge theory

Associating to automorphic forms:

global Langlands program.