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Problem 1 [17 pts.]

(a)

 [1 pt.] Correct answer: True

 [3 pts.] Justification

 Minimal polynomial of \zeta\_{16} over Q is \Phi\_{16}(x)

 Degree of \Phi\_{16}(x) is size of (Z/16Z)^{\times}

 Size of (Z/16Z)^{\times} = 8 since this is the number of odd integers between 0 and 15

(b)

 [1 pt.] Correct answer: False

 [3 pts.] Justification - give a reasonable explanation

(c)

 [1 pt.] Correct answer: False

 [2 pts.] Justification

 Irreducible polynomial of degree d admits zero in extension of degree m \implies d divides m

 3 does not divide 26

(d)

 [1 pt.] Correct answer: False

 [1 pt.] Justification - Lecture 31 Example 1

(e)

 [1 pt.] Correct answer: True

 [1 pt.] Justification - Lecture 33 Proposition 1

(f)

 [1 pt.] Correct answer: True

 [1 pt.] Justification - Q(\sqrt{2}) has characteristic 0 (which is not 2)

Problem 2 [8 pts.]

 [2 pts.] Show that Q(\sqrt{2},\sqrt{3}) is a degree 4 extension of Q

 [3 pts.] Show that Q(\sqrt{2},\sqrt{3})/Q is a Galois extension

 [3 pts.] Show that the Galois group G is isomorphic to Z/2Z \times Z/2Z

Problem 3 [8 pts.]

(a) \sqrt{2}+2\sqrt{3} in Q(\sqrt{2},\sqrt{3})

 [1 pt.] Write the Galois conjugates of \sqrt{2}+2\sqrt{3}

 \pm \sqrt{2} \pm 2 \sqrt{3}

 [3 pts.] Compute the minimal polynomial to be x^4-28x^2+100

(b) \zeta\_6+\zeta\_6^{-1} in Q(\zeta\_6)

 [2 pts.] Gal(Q(\zeta\_6)/Q) \iso (Z/6Z)^{\times} \iso Z/2Z generated by \zeta\_6 \mapsto \zeta\_6^{-1}

 [2 pts.] Compute minimal polynomial to be x-(\zeta\_6+\zeta\_6^{-1})

 \zeta\_6+\zeta\_6^{-1} is fixed by action of Galois group

 (Through various means one can show that \zeta\_6+\zeta\_6^{-1}=1)

Note: No pts. awarded for (a)/(b) if Galois orbit method is not used

Problem 4 [5 pts.]

Let L/k be finite Galois, K/k a subextension, and \sigma \in Gal(L/k)

Let \tau \in Gal(L/\sigma(K)) and \alpha \in K

 [1 pt.] Consider \sigma^{-1}\tau\sigma

 [3 pts.] Show that \sigma^{-1}\tau\sigma \in Gal(L/K)

 (\sigma^{-1}\tau\sigma)(\alpha)

 =(\sigma^{-1}\tau)(\sigma\alpha)

 =(\sigma^{-1})(\sigma\alpha) since \tau fixes \sigma(K)

 =\alpha

 [1 pt.] Reverse the argument to get Gal(L/\sigma(K))=\sigma Gal(L/K) \sigma^{-1}

Problem 5 [6 pts.]

 Show that Gal(L/K\_1K\_2) \subset H\_1 \cap H\_2

 [1 pt.] Statement

 [1 pt.] K\_1K\_2 contains K\_1 and K\_2

 [1 pt.] Fixing K\_1K\_2 implies fixing K\_1 and K\_2

 Show that H\_1 \cap H\_2 \subset Gal(L/K\_1K\_2)

 [1 pt.] Statement

 [2 pts.] Fixing K\_1,K\_2 implies fixing K\_1K\_2

 Every element of element of K\_1K\_2 is a k-rational function in elements of K\_1 and K\_2

 Galois elements are compatible with polynomial operations over k

Problem 6 [11 pts.]

 Show reflexivity

 [1 pt.] Statement

 [1 pt.] Justification

 Show symmetry

 [1 pt.] Statement

 [1 pt.] Justification

 Show transitivity

 [1 pt.] Statement

 Justification

 Suppose (a\_1,b\_1)\simeq(a\_2,b\_2) and (a\_2,b\_2)\simeq(a\_3,b\_3)

 [1 pt.] This gives a\_1b\_2=a\_2b\_1, a\_2b\_3=a\_3b\_2, and b\_1,b\_2,b\_3 nonzero

 [1 pt.] Multiply both sides of a\_1b\_2=a\_2b\_1 by b\_3

 [1 pt.] Use a\_2b\_3=a\_3b\_2 to rewrite things

 [3 pts.] Cancel off b\_2 using the fact that b\_2 is nonzero and R is an integral domain

 Conclude that a\_1b\_3=a\_3b\_1 and hence (a\_1,b\_1)\simeq(a\_3,b\_3)

Problem 7 [20 pts.]

(a)

Addition (A)

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 [4 pts.] A is well-defined

 A is commutative

 [1 pt.] Statement

 [1 pt.] Justification

 [1 pt.] 0/1 is the identity

 [1 pt.] a/b has A inverse (-a)/b=a/(-b)

 No pts. if simply called -a/b

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Multiplication (M)

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 [3 pts.] M is well-defined

 M is commutative

 [1 pt.] Statement

 [1 pt.] Justification

 [1 pt.] 1/1 is the identity

 [1 pt.] a/b nonzero has M inverse b/a

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 [2 pts.] Distributive property

(b)

Let \phi: R \to K(R) be the map r \mapsto r/1

 [2 pts.] Show that \phi is a ring homomorphism

 [1 pt.] Show that \phi is injective

Problem 8 [10 pts.]

 Give a comprehensive outline of an argument analogous to the one for Homework 7 Problem 9

 Carefully note where you need to use results for a general UFD (and not just Z)

Problem 9 [9 pts.]

 Show that if f(x) is (monic) irreducible in R[x] for R a UFD then f(x) is irreducible in K(R)[x] -- easiest to prove the contrapositive

 [1 pt.] Statement

 [1 pt.] Let f(x)=g(x)h(x) be a monic factorization of f(x) in K(R)[x]

 [2 pts.] Use Problem 8 to show g(x),h(x) are in K(R)[x]

 Conclude that f(x) is reducible in R[x]

 Find suitable k,f(x) and show that they have the desired properties

 [1 pt.] Clearly define k and f(x)

 k:=F\_p(y) and f(x):=x^p-y

 Show that f(x) is inseparable

 [1 pt.] State that it is equivalent to show f'(x)=0

 [1 pt.] Compute

 Show that f(x) is irreducible

 [1 pt.] x^p-y is irreducible in (F\_p[y])[x] by Homework 9 Problem 3

 [1 pt.] x^p-y is irreducible in (F\_p(y))[x]=k[x] by the above