Algebra II Homework 3 Commentary

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This document is intended to be a resource for you, the student. I highly encourage you to read it carefully, though of course you should first skip to the parts you feel are most relevant to you. When in doubt, V and W denote vector spaces over a fixed field k and $T : V \to W$ a k-linear map.

Problem 1

Part (a)

The problem doesn't clearly state this, but we have $I \subseteq k[x]$ an ideal generated by $q(x)$.

- It's certainly true that nontrivial factors of $q(x)$ give rise to zero divisors in $k[x]/I$. But there's more to a field than just having no zero divisors (i.e., not every integral domain is a field).
- The language here is important – $q(x)$ can be irreducible while I can be maximal (but not the other way around!).
- In a commutative ring, maximal ideals are always prime (look this up if you don't know the definition). The converse is only true for special rings.

Parts (b)-(d)

- For part (b), make sure you can clearly distinguish between the notions of algebraic and geometric multiplicity of an eigenvalue $\lambda \in \mathbb{C}$ of $A \in M_n(\mathbb{C})$. The algebraic multiplicity is the number of times $t - \lambda$ appears in the factorization of the characteristic polynomial $p_A(t) \in \mathbb{C}[t]$ into linear factors. The geometric multiplicity is the dimension of the eigenspace of λ. These need not be the same.
- For part (c), note that the zero matrix does not provide a counterexample since every nonzero vector is an eigenvector of 0 for the zero matrix.
- For part (d), it's fine to cite the Rank-Nullity Theorem.

Problem 2

- A nonempty subset $S \subseteq V$ automatically contains 0 and additive inverses ("negatives") if it is closed under addition and scaling. Here are the steps to see this.
  1. Given $v \in S$, S contains $-v$ since $-v = -1 \cdot v$.
  2. Since S is nonempty, it contains some v and hence $0 = -v + v$.

- We can't talk about multiplicative inverses here because we don't know what it means to multiply vectors. To avoid confusion, it is best to use additive and multiplicative notation where and when appropriate.
- There is a **big** difference between sums and unions of spans. To begin with, the union of spans might not even be a vector space. Think of the union of the x- and y-axes in $\mathbb{R}^2$.
- One point that threw a lot of students off is that we are simply re-indexing. You can think of an element of $\mathrm{Span}(v_1, \dots, v_m) + \mathrm{Span}(w_1, \dots, w_n)$ as $(a_1 v_1 + \cdots + a_m v_m) + (b_1 w_1 + \cdots + b_n w_n)$ for some $a_1, \dots, a_m, b_1, \dots, b_n \in k$. Equivalently, you can think of it as an element $c_1 v_1 + \cdots + c_m v_m + c_{m+1} w_1 + \cdots + c_{m+n} w_n$ for some $c_1, \dots, c_{m+n} \in k$. That is,

$$c_i = \begin{cases} a_i, & i \leq m, \\ b_{m-i}, & i > m. \end{cases}$$

  This is what it means to say

$$\mathrm{Span}(v_1, \dots, v_m) + \mathrm{Span}(w_1, \dots, w_n) = \mathrm{Span}(v_1, \dots, v_m, v_1, \dots, v_n).$$

Problem 3

- Diagonal and upper triangular matrices aren't the same thing.
- For writing matrices, note that you have vertical and diagonal dots in addition to horizontal ones. Matrices are typically written

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

  The code for this is

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\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix}
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- For (a) $\implies$ (b), you need to describe the basis $\{v_1, \dots, v_n\}$ and not just say it exists. It is also best to work with an explicit basis of k^n, namely the standard one.
- For (b) $\implies$ (c), many people forgot to specify that the choice of basis is the same. Given general $v \in \mathrm{Span}(v_1, \dots, v_m)$ (for some $m \leq n$), it is not necessarily the case that $T_A(v) =$

$\sum_{i \leq j} b_{ij} v_i$ for some j. What is true is that we can write $v = a_1 v_1 + \cdots + a_i v_i$ and so

$$\begin{aligned} T_A(v) &= T_A\left(\sum_{j=1}^{m} a_j v_j\right) \\ &= \sum_{j=1}^{m} a_j T_A(v_j) \\ &= \sum_{j=1}^{m} a_j \left(\sum_{i \leq j} b_{ij} v_i\right) \\ &= \sum_{j=1}^{m} c_j v_j \\ &\in \operatorname{Span}(v_1, \ldots, v_m). \end{aligned}$$

Think carefully about what each coefficient c_j above looks like.

Problem 4

Part (a)

- The fact that T maps $W \leq V$ to W does not mean that T maps a complement of W to a complement of W. Indeed, think of the projection map sending all of V to W.
- You need to be careful with inputs since $\overline{T}$ only makes sense on cosets and T only makes sense on vectors in V.
- It's true that T fits into a commutative diagram, as can be seen from

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & V \arrow[d, "\pi"] \\ V/W \arrow[r, "\overline{T}"'] & V/W \end{tikzcd}$$

  for $\pi : V \twoheadrightarrow V/W$ the projection map given by $v \mapsto v + W$. This doesn't mean that $\overline{T}$ is given by a composition. Indeed, one must check that $\overline{T}$ is both well-defined and k-linear.

Part (b)

- Technically this problem isn't true as stated since eigenvectors are required to be nonzero.
- Saying $T(v) = \lambda v + W$ makes no sense since the LHS is a vector while the RHS is a coset.
- Given $a \in k$ and $w \in W$, it's **not** true that $a(v + W) = av + aW$ as sets. By definition, $a(v + W) = av + W$ in the quotient space V/W. If $a = 0$ then the set $av + aW$ is simply $\{0\}$ while $0(v + W) = W$. Anyone confused about this should review the quotient construction.

Part (c)

- A big part of this problem is that an eigenbasis of T_A need not exist. If you look closely at the solution to this problem then you will see that we find eigenvectors of quotient spaces and not the whole space.

- Related to the above, eigenvectors of T_A and $\overline{T_A}$ are very different things. For starters, the two maps aren't even defined on the same vector space.

Problem 5

- As in the problem statement, let $V_1, \ldots, V_r$ be k-vector spaces with bases $\mathcal{B}_1, \ldots, \mathcal{B}_r$. How do we use this to get a basis for $V := V_1 \times \cdots \times \mathcal{B}_r$? It's not $\mathcal{B}_1 + \cdots + \mathcal{B}_r$ or $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ since neither of those even makes sense. It's also not $\mathcal{B}_1 \times \cdots \times \mathcal{B}_r$ (Can you see why?). Instead, what we do is extend each $v_{i,j} \in \mathcal{B}_i$ to a vector in V by putting a bunch of zeros, giving us a new linearly independent set $\overline{\mathcal{B}_i} \subseteq V$. For example, if $\mathcal{B}_1 = \{v_{1,1}, \ldots, v_{1,n_1}\}$ then $\overline{\mathcal{B}_1} = \{w_{1,1}, \ldots, w_{1,n_1}\}$ with $w_{i,j} = (v_{1,j}, 0, \ldots, 0)$. One can then show that $\mathcal{B} := \overline{\mathcal{B}_1} \cup \cdots \cup \overline{\mathcal{B}_r}$ is a basis of V. Note that this gives us a hands-on way of checking that $\dim_k(V_1) + \cdots + \dim_k(V_r) = \dim_k(V)$.
- Remember that the data of a k-pair includes a linear map $T : V \to V$ and not just a vector space V. Many people lost points on this problem for failing to keep track of linear maps.
- Morally speaking, the "right" way to do this problem is to define an isomorphism

$$k^n \xrightarrow{\sim} k^{n_1} \times \cdots \times k^{n_r} \xrightarrow{\sim} V_1 \times \cdots \times V_r$$

  since mapping out of a space of the form k^m is something we know how to do explicitly.
- The problem isn't clear about whether we need to work the product k-pair $(V_1, T_1) \times \cdots \times (V_r, T_r)$ or $V_{A_1} \times \cdots \times V_{A_r}$. Thus, I awarded (up to 2) bonus points to anyone who showed these two pairs are isomorphic.

Problem 6

Many people lost points for failing to explain why $V_{x-\lambda_j}$ is (isomorphic to) the eigenspace for λ_j.

Problem 7

Don't just reference results from class – perform the computation.

Problem 8

Part (a)

In general, it may not be true that $r_1 i_2 + I_1 = r_1 i_2 + r_1 I_1$ and $r_1 + I_1 = r_1 + r_1 I_1$. For a concrete example, compare the sets $2 + 3\mathbb{Z}$ and $2 + 2 \cdot 3\mathbb{Z} = 2 + 6\mathbb{Z}$.

Part (b)

As a reminder, recall that for each $1 \leq k \leq n-1$ we choose $i_k \in I_k$ and $i_{n,k} \in I_n$ such that $i_k + i_{n,k} = 1$ (incidentally, many people lost points for failing to say why these elements exist).

- Many students were confused about how to describe the product $(i_1 + i_{n,1}) \cdots (i_{n-1} + i_{n,n-1})$. One way of describing it is $i_1 \cdots i_{n-1} + \alpha$ for $\alpha \in R$ an element of the ideal $(i_{n,1}, \ldots, i_{n,n-1})$ generated by the elements $i_{n,k}$. Hence, $\alpha \in I_n$ since this whole ideal is contained in I_n.

- It's **not** true that $1 = (i_1+i_{1,n}) \cdots (i_{n-1}+i_{n-1,n})$ implies $i_1 \cdots i_{n-1} = (1-i_{1,n}) \cdots (1-i_{n-1,n})$. Arithmetic doesn't work like that.

Part (c)

- Don't forget about the trivial case $n = 1$!
- Let $\varphi : R \to R/I_1 \times \cdots \times R/I_n$ be the map $r \mapsto (r + I_1, \ldots, r + I_n)$. It's **not** true that $\varphi(r_1 + \cdots + r_n) = (r_1 + I_1, \ldots, r_n + I_n)$. This is true if you can arrange that $r_i \in I_j$ for $i \neq j$, but many people incorrectly reached this condition from a miscalculation in part (b).
- In the inductive step, where we replace $R/I_1 \times \cdots \times R/I_n$ by $R/(I_1 \cap \cdots \cap I_{n-1}) \times R/I_n$ using the isomorphism $\psi : R/(I_1 \cap \cdots \cap I_{n-1}) \xrightarrow{\sim} R/I_1 \times \cdots \times R/I_{n-1}$, we technically need to check that the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/I_1 \times \cdots \times R/I_n \\ R/(I_1 \cap \cdots \cap I_{n-1}) \times R/I_n \arrow[ur, "\psi\times\mathrm{id}", "\cong"'] & \end{tikzcd}$$

  commutes, where $(\psi \times \mathrm{id})(\overline{r}, s + I_n) = (\psi(\overline{r}), s + I_n)$.

Part (d)

- We need not just isomorphism of the ideals $I_1 \cdots I_n$ and $I_1 \cap \cdots \cap I_n$ (think carefully about what that even means) but in fact equality.
- In general, products of ideals consists of sums of products of elements and not just products of elements. Concretely, given ideals $I, J \subseteq R$,

$$IJ = \left\{ \sum_{k=1}^{m} x_k y_k : x_k \in I, y_k \in J, m \in \mathbb{N} \right\}.$$

Problem 9

Define $I_j := (q_j(x)) \subseteq k[x]$ for $1 \leq j \leq m$ as well as $I := (q(x))$.

- Many people used the notation I_j without defining it.
- Many people correctly said that $I_1, \ldots, I_m$ are pairwise relatively prime but neglected to prove it. It is true that we then have $k[x]/(I_1 \cdots I_m) \cong k[x]/I_1 \times \cdots \times k[x]/I_m$. The final step is to show that $I = I_1 \cdots I_m$.
- Many students lost points for failing to mention compatibility of multiplication-by-x maps. Remember that we want an isomorphism of pairs and not just vector spaces.