\ov = \overline (overline)

\iso = \cong (isomorphism)

Problem 1 [11 pts.]

(a)

 [1 pt.] Correct answer: True

 [2 pts.] Justification

(b)

 [1 pt.] Correct answer: False

 [1 pt.] Justification -- compare mult. by x-1 on both sides

(c)

 [1 pt.] Correct answer: False

 [1 pt.] Justification -- Homework 4 Problem 4(c)

(d)

 [1 pt.] Correct answer: False

 [1 pt.] Justification -- Homework 4 Problem 7

(e)

 [1 pt.] Correct answer: True

 [1 pt.] Justification -- every Euclidean domain is a PID hence a UFD

Problem 2 [11 pts.]

Fix a,b\in R and let \phi: R/(a)\to R/(a) denote mult. by b

 Allowed to assume (without statement or justification) that (a),(b) are relatively prime iff there exist x,y\in R such that ax+by=1

 (a)-->(b)

 [1 pt.] Write 1=ax+by

 Show that y+(a) is inverse of b+(a)

 [1 pt.] Statement

 [1 pt.] Justification (computation)

 (b)-->(c)

 Show \phi is injective

 [1 pt.] Pick something in \ker\phi

 [1 pt.] Write things out in terms of cosets

 [1 pt.] Multiply by the inverse of b+(a)

 Show \phi is surjective

 [1 pt.] Describe preimage

 [1 pt.] Justification (computation)

 Alternative approach: explicitly construct inverse of \phi

 (c)-->(a)

 [1 pt.] Choose c\in R such that \phi(c+(a))=1+(a)

 [2 pts.] Write 1 as sum of elements of (a) and (b)

Problem 3 [11 pts.]

Let \phi be the mult. by f(x) map on k[x]/(q(x))

 (a)-->(b)

 [1 pt.] Choose g(x)\in k[x] such that f(x)g(x)=q(x)

 Show that \phi(\ov{g(x)})=0

 [1 pt.] Statement

 [1 pt.] Justification

 Show that \ov{g(x)} is nonzero in k[x]/(q(x))

 [1 pt.] Statement

 [1 pt.] Justification

 (b)-->(a)

 Show that (f(x)) and (q(x)) are not relatively prime

 [1 pt.] Statement

 [1 pt.] Justification -- Problem 2

 Show that gcd(f(x),q(x)) \neq 1

 [1 pt.] Statement

 [1 pt.] Justification -- (f(x))+(q(x))=(gcd(f(x),q(x)))

 Show that f(x) divides q(x)

 [1 pt.] Statement

 [1 pt.] Justification -- f(x) is irreducible

Problem 4 [16 pts.]

We write down the details for the cycle (a)-->(b)-->(c)-->(d)-->(a)

Let f\_1|...|f\_r be the invariant factors of A\in M\_n(k)

 (a)-->(b)

 [1 pt.] State that min\_A(x) divides ch\_A(x)

 (b)-->(c)

 Show that \lambda is a zero of some f\_j(x)

 [1 pt.] Statement

 [1 pt.] Justification -- ch\_A=f\_1...f\_r

 [1 pt.] State that (x-\lambda) divides some f\_j(x)

 [1 pt.] State that multiplication by (x-\lambda) is not injective on V\_{f\_j) (c.f. Problem 3)

 [1 pt.] State that multiplication by (x-\lambda) on V\_{f\_1}\times\cdots\times V\_{f\_r} is not injective

 Show that action of T\_A-\lambda\id on V\_A is not injective (hence T\_A-\lambda\is not an isomorphism)

 [1 pt.] Statement

 [1 pt.] Justification -- V\_A\iso V\_{f\_1}\times V\_{f\_r} with T\_A corresponding to multiplication by x

 (c)-->(d)

 [1 pt.] Briefly explain why T-\lambda\id is not injective -- Rank-Nullity Thm

 [1 pt.] Choose v nonzero in the kernel of T\_A-\lambda\id

 [1 pt.] Show that Av-\lambda v=0 -- i.e., \lambda is an eigenvalue of A with eigenvector v

 (d)-->(a)

 [1 pt.] State that T\_A-\lambda\id is not injective on V\_A

 [1 pt.] State that x-\lambda is not injective on V\_{f\_1}\times\cdots\times V\_{f\_r}

 [1 pt.] State that x-\lambda is not injective on V\_{f\_r}

 Show that (x-\lambda) divides min\_A(x)=f\_r(x)

 [1 pt.] Statement

 [1 pt.] Justification -- Problem 3

Problem 5

Let A\in M\_n(k) and p(x)\in k[x]

Let V\_A \iso V\_{f\_1}\times\cdots\times V\_{f\_r} for (unique) monic f\_1|...|f\_r in k[x], so f\_r(x)=min\_A(x)...

Note that p(A) is the matrix of p(T\_A) (with respect to the standard basis on k^n)

 Students are not required to say anything about this

 [1 pt.] State that p(T\_A) acting on V\_A corresponds to p(x) acting on V\_{f\_1}\times\cdots\times V\_{f\_r} by multiplication

 (a)-->(b)

 [1 pt.] State that p(T\_A) acts trivially on V\_A

 [1 pt.] State that p(x) acts trivially on V\_{f\_1}\times\cdots\times V\_{f\_r}

 [1 pt.] State that p(x) acts trivially on V\_{f\_r}

 [1 pt.] Explain why f\_r(x)=min\_A(x) divides p(x) (c.f. Problem 3)

 (b)-->(a)

 Show that p(x) acts trivially on each V\_{f\_i}

 [1 pt.] Statement

 [1 pt.] Justification

 [1 pt.] State that p(x) acts trivially on V\_{f\_1}\times\cdots\times V\_{f\_r}

 [1 pt.] State that p(T\_A) acts trivially on V\_A

 [1 pt.] State that p(A)=0 in M\_n(k)

 [1 pt.] Explain why ch\_A(A)=0 -- min\_A(x) divides ch\_A(x) by definition

While it's true that many of the steps for (a)-->(b) are reversible in a certain sense, work needs to go into showing this.

Problem 6 [6 pts.]

 A is invertible

 [1 pt.] \iff T\_A: k^n\to k^n is an isomorphism

 [2 pts.] \iff 0 is not a zero of ch\_A(x) by Problem 4

 [2 pts.] \iff det(A) \neq 0 by definition

 [1 pt.] for mentioning that everything is reversible

Alternative approach is to directly use Cayley-Hamilton Thm

 Minimality (in terms of degree) of the minimal polynomial is key

Problem 7 [12 pts.]

 [1 pt.] State that x^2+1 factors in F\_5[x] as (x-2)(x-3)

 Show that all possibilites for A are diagonalizable

 [1 pt.] Statement

 [1 pt.] Justification -- a matrix is diagonalizable iff its minimal polynomial factors completely as a product of distinct linear factors

Invariant Factors ch\_A(x) det(A) tr(A)

x^2+1,x^2+1 x^4+2x^2+1 1 0

x-2,x-2,x^2+1 x^4-4x^3-4x+4 4 4

x-3,x-3,x^2+1 x^4+4x^3+4x+4 4 -4=1

 [1 pt.] for each correct invariant factor, determinant, and trace

 [-1 pt.] overall for sign errors on the trace

 No pts. awarded if no evidence of work

 Only partial pts. awarded if interpretation of matrices is incorrect

 I decided not worry about whether students actually write down block matrices

 Lot's of potential for extra sign errors and such...

Problem 8

(a)

 Show that M^{tors} is closed under addition

 [1 pt.] Given m,n\in M^{tors}, choose nonzero r,s\in R such that rm=0=sn

 [1 pt.] Show rs(m+n)=0

 Show that rs is nonzero

 [1 pt.] Statement

 [1 pt.] Justification -- R is an integral domain

 [1 pt.] Show that M^{tors} is closed under scalar multiplication

 Show that M^{tf} is torsion-free

 [1 pt.] Let m\in M and r\in R nonzero such that r\ov{m}=0 in M^{tf}

 [1 pt.] State that rm\in M^{tors}

 [1 pt.] Show that m\in M^{tors}

 Choose s\in R nonzero such that (sr)m=s(rm)=0 in M

 Mention that sr is nonzero (since R is an integral domain)

(b)

Let \phi: M\to N be an R-module isomorphism

 Show that \phi carries M^{tors} isomorphically onto N^{tors}

 [1 pt.] State that the restriction of \phi to M^{tors} is injective

 [2 pts.] Show that \phi(M^{tors}) \subset N^{tors}

 [3 pts.] Show that N^{tors} \subset \phi(M^{tors})

 Use surjectivity of \phi

 Perform the relevant computations

 Use injectivity of \phi

 Show that \phi induces an isomorphism \ov{\phi}: M^{tf}\to N^{tf}

 [1 pt.] Give a formula for \ov{\phi}

 Show that \ov{\phi} is well-defined

 [1 pt.] Statement

 [1 pt.] Justification

 [2 pts.] Show that \ov{\phi} is injective

 [1 pt.] Show that \ov{\phi} is surjective

(c)

For convenience, assume that M = R/(a\_1)\times\cdots\times R/(a\_r)\times R^m for a\_i\in R nonzero nonunits

 Show that M^{tors} \iso R/(a\_1)\times\cdots\times R/(a\_r)

 Show that R/(a\_1)\times\cdots\times R/(a\_r) injects into M^{tors}

 [1 pt.] Statement (needs to be precise)

 [1 pt.] Justification

 Show that M^{tors} injects into R/(a\_1)\times\cdots\times R/(a\_r)

 [1 pt.] Statement (statement needs to be precise)

 [2 pts.] Justification

 [4 pts.] Show that M^{tf} \iso R^m

No need to write down explicit isomorphisms for part (c)