Problem 1 [8 pts.]

(a)

 [1 pt.] Correct answer: false

 [1 pt.] Justification

(b)

 [1 pt.] Correct answer: false

 [1 pt.] Justification (ok if no counterexample given)

 Half pts. awarded if student says true and justifies that Z/nZ is principal

(c)

 [1 pt.] Correct answer: false

 [1 pt.] Justification (must give counterexample)

(d)

 [1 pt.] Correct answer: false

 [1 pt.] Justification (must give counterexample)

Problem 2 [6 pts.]

Let I be the union of the I\_k

 [1 pt.] Identify element of I as element of some I\_k

 [1 pt.] State that pair of elements in I land in common I\_j

 [2 pts.] Show closure under addition

 Statement

 Justification

 [2 pts.] Show closure under scaling by elements of R

 Statement

 Justification

Problem 3 [8 pts.]

 [1 pt.] Address the case of the zero matrix

 [1 pt.] State that conjugacy classes (of nonzero elements!) are uniquely and completely represented by matrices in rational canonical form

 No pts. off if failure to qualify nonzero

 Identify the diagonal RCFs

 [1 pt.] State that diagonal RCFs are allowed and correspond to characteristic matrices for a pair of degree 1 monic polynomials

 [2 pts.] Deduce that both polynomials must be the same

 Statement

 Justification

 [1 pt.] List the possibilities (3 total)

 Identify the non-diagonal RCFs

 [1 pt.] State that there is only one type of non-diagonal RCF in the 2x2 case (which corresponds characteristic matrix of deg 2 monic polynomial)

 [1 pt.] List the possibilities (9 total)

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Characteristic Matrices

 1x1:

 \*

 2x2:

 0 \*

 1 \*

RCFs

 \* 0

 0 \*

 with \* matching

 TOTAL #: 3

 0 \*

 1 \*

 TOTAL #: 9

 0 0

 0 0 (not an RCF technically)

 TOTAL #: 1

GRAND TOTAL: 13

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Problem 4 [10 pts.]

Let L/k be a field extension and B \in M\_n(k)

We distinguish between invariant factors over k and over L

(a)

 [2 pts.] RCF over k --> RCF over L

 RCF over L --> RCF over k

 [1 pt.] Show that invariant factors of B over L lie in k[x]

 [3 pts.] Show that if f,g \in k[x] with g|f in L[x] then g|f in k[x]

 Method 1: induction

 Base case

 Inductive case

 Calculation

 Method 2: division algorithm (DA)

 Perform DA over k[x]

 Compare with DA over L[x]

 Conclude remainder vanishes

 [1 pt.] Deduce that invariant factors of B over L work over k as well

(b)

 [1 pt.] Appeal to (a)

(c)

 [1 pt.] Translate to statement about RCFs

 [1 pt.] Appeal to (b) (implicitly or explicitly)

Problem 5 [10 pts.]

Let \phi be the map from R to S

(a) --> (b)

 [1 pt.] Appeal to Homework 3 Problem 8

(b) --> (c)

 [1 pt.] Give brief justification

(c) --> (a)

 [1 pt.] Choose elements r\_1,...,r\_n\in R such that \phi(r\_1)=e\_1,...,\phi(r\_n)=e\_n

 [1 pt.] Take the sum r:=r\_1+\cdots+r\_n

 [1 pt.] Deduce that r-1\in\ker(\phi)

 [1 pt.] Deduce that r-1=a for some a\in I\_1\cap\cdots\cap I\_n

 Deduce that I\_i and I\_j are relatively prime for i\neq j

 [2 pts.] Find elements x\in I\_1 and y\in I\_2 such that x+y=1

 [2 pts.] Correctly explain what to do for general case n>2

Problem 6 [6 pts.]

 [1 pt.] State that k-pair is isomorphic to V\_{f\_1}\times\cdots V\_{f\_m} (rational canonical form)

 f\_1,...,f\_m \in k[x] unique non-constant monics

 f\_1|f\_2|...|f\_m

 [1 pt.] Factor each f\_i into product of monic irreducibles in k[x], with exponents to keep factors distinct

 Deduce that V\_{f\_i} is isomorphic to product of k-pairs of form V\_{q^e} for q\in k[x] monic irreducible

 [1 pt.] State that the different terms in the factorization are relatively prime

 [1 pt.] Use Homework 3 Problem 9 (implicitly or explicitly)

 Show uniqueness up to permutation

 [1 pt.] Appeal to uniqueness of rational canonical form

 [1 pt.] Appeal to uniqueness of prime factorization

 [-1 pt.] Trying to "recombine" factors, resulting e.g. in false equivalence of k-pairs between V\_x\times V\_x and V\_{x^2}

Problem 7 [11 pts.]

Let A\in M\_n(k) with invariant factors f\_1|f\_2|...|f\_m

 A diagonalizable --> f\_m(x)=(x-\lambda\_1)\cdots(x-\lambda\_r) separable (i.e., distinct roots)

 Write V\_A \iso V\_{x-\lambda}^{e\_1}\times\cdots\times V\_{x-\lambda\_r}^{e\_r} for \lambda\_1,...,\lambda\_r distinct

 [1 pt.] Use Homework 3 Problem 6

 [1 pt.] Collect like terms

 Deduce that f\_m(x)=(x-\lambda\_1)^{a\_1}\cdots(x-\lambda\_r)^{a\_r}

 [1 pt.] Statement

 [1 pt.] Justification -- key is construction from Problem 6

 Deduce that each a\_i=1

 [1 pt.] Statement

 [1 pt.] Justification -- key is uniqueness part of Problem 6

 f\_m(x)=(x-\lambda\_1)\cdots(x-\lambda\_r) separable --> A diagonalizable

 Deduce that each f\_i is a product of linear factors

 [1 pt.] Statement

 [1 pt.] Justification -- f\_1|...|f\_m

 Deduce that V\_A \iso V\_{x-\alpha\_1}\times\cdots\times V\_{x-\alpha\_n} for some \alpha\_1,...,\alpha\_n\in k

 [1 pt.] Statement

 [1 pt.] Justification

 [1 pt.] Conclude that A is diagonalizable

Problem 8 [8 pts.]

(a)

 [1 pt.] Use Homework 3 Problem 5

 [1 pt.] Use Homework 3 Problem 7

 [1 pt.] Regroup to get distinct roots

 [1 pt.] Use Homework 3 Problem 9

 [1 pt.] Conclude that f\_m(x) is product of linear factors to get (i)-->(ii)

 [1 pt.] Mention that all steps are reversible to get (ii)-->(i)

(b)

 [1 pt.] State that JCF of A yields product decomposition of V\_A akin to Problem 6

 [1 pt.] Use uniqueness part of Problem 6 to conclude that JCF of A is unique

Problem 9 [18 pts.]

Let the norm conditions be:

 (a) N(a)=0 iff a=0

 (b) N(ab) \leq N(a)N(b) (sub-multiplicative)

 (c) Division algorithm -- important that a \neq 0

Part 1: Z

 [1 pt.] Check (a) -- By construction

 [1 pt.] Check (b) -- Use multiplicativity of absolute value

 [1 pt.] Check (c) -- Need to mention something about the negative case

Part 2: k[x]

 [2 pts.] Check (a)

 State N(0)=0

 Check N(f)=0 --> f=0

 [2 pts.] Check (b)

 Use degree-sum formula

 Show multiplicativity with correct arithmetic

 [3 pts.] Check (c)

Part 3: Z[i]

 [2 pts.] Check (a)

 State N(0)=0

 Check N(f)=0 --> f=0 (no pts. if no explanation)

 [2 pts.] Check (b) -- Show multiplicativity with correct arithmetic

 Check (c) -- Let a,b\in Z[i] with a \neq 0

 [1 pt.] Consider b/a as element of Q[i]

 [1 pt.] Argue that we can choose q\in Z[i] such that N(b/a-q)\leq 1/2 (notice that N extends to all of C)

 One approach is to consider q:=[c]+[d]i for b/a=c+di and [.] the nearest integer function

 Another approach is to use geometry

 [1 pt.] Multiply through by N(a) to get N(b-aq)\leq N(a)/2

 [1 pt.] Take remainder r=b-aq

Problem 10 [8 pts.]

Let I be a nonzero ideal of the Euclidean domain (R,N)

 [1 pt.] Select element a in I with minimal positive norm

 [1 pt.] Given element b in I, apply division algorithm to get q,r with N(r)<N(a) such that b=qa+r

 [2 pts.] Deduce that r is in I

 Statement

 Justification

 [2 pts.] Deduce that N(r)=0

 Statement

 Justification

 [1 pt.] Deduce that r=0

 [1 pt.] Conclude that I=(a)

 Bonus! [1 pt.] Conclude that R is a PID (must also address zero ideal)