Problem 1 [8 pts.]

(a)

[1 pt.] Correct answer: false

[1 pt.] Justification

(b)

[1 pt.] Correct answer: false

[1 pt.] Justification (ok if no counterexample given)

Half pts. awarded if student says true and justifies that Z/nZ is principal

(c)

[1 pt.] Correct answer: false

[1 pt.] Justification (must give counterexample)

(d)

[1 pt.] Correct answer: false

[1 pt.] Justification (must give counterexample)

Problem 2 [6 pts.]

Let I be the union of the I\_k

[1 pt.] Identify element of I as element of some I\_k

[1 pt.] State that pair of elements in I land in common I\_j

[2 pts.] Show closure under addition

Statement

Justification

[2 pts.] Show closure under scaling by elements of R

Statement

Justification

Problem 3 [8 pts.]

[1 pt.] Address the case of the zero matrix

[1 pt.] State that conjugacy classes (of nonzero elements!) are uniquely and completely represented by matrices in rational canonical form

No pts. off if failure to qualify nonzero

Identify the diagonal RCFs

[1 pt.] State that diagonal RCFs are allowed and correspond to characteristic matrices for a pair of degree 1 monic polynomials

[2 pts.] Deduce that both polynomials must be the same

Statement

Justification

[1 pt.] List the possibilities (3 total)

Identify the non-diagonal RCFs

[1 pt.] State that there is only one type of non-diagonal RCF in the 2x2 case (which corresponds characteristic matrix of deg 2 monic polynomial)

[1 pt.] List the possibilities (9 total)

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Characteristic Matrices

1x1:

\*

2x2:

0 \*

1 \*

RCFs

\* 0

0 \*

with \* matching

TOTAL #: 3

0 \*

1 \*

TOTAL #: 9

0 0

0 0 (not an RCF technically)

TOTAL #: 1

GRAND TOTAL: 13

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Problem 4 [10 pts.]

Let L/k be a field extension and B \in M\_n(k)

We distinguish between invariant factors over k and over L

(a)

[2 pts.] RCF over k --> RCF over L

RCF over L --> RCF over k

[1 pt.] Show that invariant factors of B over L lie in k[x]

[3 pts.] Show that if f,g \in k[x] with g|f in L[x] then g|f in k[x]

Method 1: induction

Base case

Inductive case

Calculation

Method 2: division algorithm (DA)

Perform DA over k[x]

Compare with DA over L[x]

Conclude remainder vanishes

[1 pt.] Deduce that invariant factors of B over L work over k as well

(b)

[1 pt.] Appeal to (a)

(c)

[1 pt.] Translate to statement about RCFs

[1 pt.] Appeal to (b) (implicitly or explicitly)

Problem 5 [10 pts.]

Let \phi be the map from R to S

(a) --> (b)

[1 pt.] Appeal to Homework 3 Problem 8

(b) --> (c)

[1 pt.] Give brief justification

(c) --> (a)

[1 pt.] Choose elements r\_1,...,r\_n\in R such that \phi(r\_1)=e\_1,...,\phi(r\_n)=e\_n

[1 pt.] Take the sum r:=r\_1+\cdots+r\_n

[1 pt.] Deduce that r-1\in\ker(\phi)

[1 pt.] Deduce that r-1=a for some a\in I\_1\cap\cdots\cap I\_n

Deduce that I\_i and I\_j are relatively prime for i\neq j

[2 pts.] Find elements x\in I\_1 and y\in I\_2 such that x+y=1

[2 pts.] Correctly explain what to do for general case n>2

Problem 6 [6 pts.]

[1 pt.] State that k-pair is isomorphic to V\_{f\_1}\times\cdots V\_{f\_m} (rational canonical form)

f\_1,...,f\_m \in k[x] unique non-constant monics

f\_1|f\_2|...|f\_m

[1 pt.] Factor each f\_i into product of monic irreducibles in k[x], with exponents to keep factors distinct

Deduce that V\_{f\_i} is isomorphic to product of k-pairs of form V\_{q^e} for q\in k[x] monic irreducible

[1 pt.] State that the different terms in the factorization are relatively prime

[1 pt.] Use Homework 3 Problem 9 (implicitly or explicitly)

Show uniqueness up to permutation

[1 pt.] Appeal to uniqueness of rational canonical form

[1 pt.] Appeal to uniqueness of prime factorization

[-1 pt.] Trying to "recombine" factors, resulting e.g. in false equivalence of k-pairs between V\_x\times V\_x and V\_{x^2}

Problem 7 [11 pts.]

Let A\in M\_n(k) with invariant factors f\_1|f\_2|...|f\_m

A diagonalizable --> f\_m(x)=(x-\lambda\_1)\cdots(x-\lambda\_r) separable (i.e., distinct roots)

Write V\_A \iso V\_{x-\lambda}^{e\_1}\times\cdots\times V\_{x-\lambda\_r}^{e\_r} for \lambda\_1,...,\lambda\_r distinct

[1 pt.] Use Homework 3 Problem 6

[1 pt.] Collect like terms

Deduce that f\_m(x)=(x-\lambda\_1)^{a\_1}\cdots(x-\lambda\_r)^{a\_r}

[1 pt.] Statement

[1 pt.] Justification -- key is construction from Problem 6

Deduce that each a\_i=1

[1 pt.] Statement

[1 pt.] Justification -- key is uniqueness part of Problem 6

f\_m(x)=(x-\lambda\_1)\cdots(x-\lambda\_r) separable --> A diagonalizable

Deduce that each f\_i is a product of linear factors

[1 pt.] Statement

[1 pt.] Justification -- f\_1|...|f\_m

Deduce that V\_A \iso V\_{x-\alpha\_1}\times\cdots\times V\_{x-\alpha\_n} for some \alpha\_1,...,\alpha\_n\in k

[1 pt.] Statement

[1 pt.] Justification

[1 pt.] Conclude that A is diagonalizable

Problem 8 [8 pts.]

(a)

[1 pt.] Use Homework 3 Problem 5

[1 pt.] Use Homework 3 Problem 7

[1 pt.] Regroup to get distinct roots

[1 pt.] Use Homework 3 Problem 9

[1 pt.] Conclude that f\_m(x) is product of linear factors to get (i)-->(ii)

[1 pt.] Mention that all steps are reversible to get (ii)-->(i)

(b)

[1 pt.] State that JCF of A yields product decomposition of V\_A akin to Problem 6

[1 pt.] Use uniqueness part of Problem 6 to conclude that JCF of A is unique

Problem 9 [18 pts.]

Let the norm conditions be:

(a) N(a)=0 iff a=0

(b) N(ab) \leq N(a)N(b) (sub-multiplicative)

(c) Division algorithm -- important that a \neq 0

Part 1: Z

[1 pt.] Check (a) -- By construction

[1 pt.] Check (b) -- Use multiplicativity of absolute value

[1 pt.] Check (c) -- Need to mention something about the negative case

Part 2: k[x]

[2 pts.] Check (a)

State N(0)=0

Check N(f)=0 --> f=0

[2 pts.] Check (b)

Use degree-sum formula

Show multiplicativity with correct arithmetic

[3 pts.] Check (c)

Part 3: Z[i]

[2 pts.] Check (a)

State N(0)=0

Check N(f)=0 --> f=0 (no pts. if no explanation)

[2 pts.] Check (b) -- Show multiplicativity with correct arithmetic

Check (c) -- Let a,b\in Z[i] with a \neq 0

[1 pt.] Consider b/a as element of Q[i]

[1 pt.] Argue that we can choose q\in Z[i] such that N(b/a-q)\leq 1/2 (notice that N extends to all of C)

One approach is to consider q:=[c]+[d]i for b/a=c+di and [.] the nearest integer function

Another approach is to use geometry

[1 pt.] Multiply through by N(a) to get N(b-aq)\leq N(a)/2

[1 pt.] Take remainder r=b-aq

Problem 10 [8 pts.]

Let I be a nonzero ideal of the Euclidean domain (R,N)

[1 pt.] Select element a in I with minimal positive norm

[1 pt.] Given element b in I, apply division algorithm to get q,r with N(r)<N(a) such that b=qa+r

[2 pts.] Deduce that r is in I

Statement

Justification

[2 pts.] Deduce that N(r)=0

Statement

Justification

[1 pt.] Deduce that r=0

[1 pt.] Conclude that I=(a)

Bonus! [1 pt.] Conclude that R is a PID (must also address zero ideal)