Problem 1 [8 pts.]

(a)

[1 pt.] Correct answer: true

[1 pt.] Justification

(b)

[1 pt.] Correct answer: false

[1 pt.] Justification

(c)

[1 pt.] Correct answer: true

[1 pt.] Justification

(d)

[1 pt.] Correct answer: true

[1 pt.] Justification

Problem 2 [4 pts.]

Let W\_1,W\_2 \leq V and S={v\_1,...,v\_m},T={w\_1,...,w\_n} subsets of V

Part 1: W\_1+W\_2 \leq V

[1 pt.] Closure under addition

[1 pt.] Closure under scaling

Part 2: Span(S)+Span(T)=Span(S \cup T)

[1 pt.] Write things in terms of linear combinations

[1 pt.] Explain why we have equality

Problem 3 [10 pts.]

[3 pts.] (a) --> (b)

[3 pts.] (b) --> (c)

[3 pts.] (c) --> (a)

[1 pt.] Full logical equivalence demonstrated

Problem 4 [22 pts.]

(a)

[2 pts.] Show that \overline{T} is well-defined

State precisely what this means

Check the condition

[2 pts.] Show that \overline{T} is k-linear

Respects vector sums

Respects scaling

(b)

[1 pt.] Rewrite (i)

[3 pts.] Rewrite (ii)

[1 pt.] Show (i) \implies (ii)

[1 pt.] Show (ii) \implies (i)

(c)

[1 pt.] Use Problem 3 to translate to finding basis {v\_1,...,v\_n} of k^n such that T\_A(W\_i) \leq W\_i for every i with W\_i := Span(v\_1,...,v\_i)

Base case (i=1)

[1 pt.] Choose v\_1 an eigenvector of T\_A (use HW Problem 6)

[1 pt.] Show that T\_A(Span(v\_1)) \leq Span(v\_1)

Inductive case (i>1)

[1 pt.] Invoke inductive hypothesis to choose v\_1,...,v\_{i-1} linearly independent with T\_A(W\_{i-1}) \leq W\_{i-1}

[1 pt.] Choose v\_i such that v\_i+W\_{i-1} is an eigenvector of \overline{T\_A}: k^n/W\_{i-1} \to k^n/W\_{i-1} (use HW 2 Problem 6)

[4 pts.] Show that T\_A(W\_i) \leq T\_A(W\_i)

Statement

Justification

Use (b) to show that T\_A(v\_i)=av\_i+w for some a in k and w in W\_{i-1}

Use Problem 2 to say that W\_i=Span(v\_i)+W\_{i-1}

Use inductive hypothesis

[3 pts.] Show that v\_1,...,v\_i are linearly independent

Statement

Justification

Use that v\_i+W\_{i-1} is nonzero in k^n/W\_{i-1} to say v\_i not in W\_{i-1}

Use inductive hypothesis

Problem 5 [7 pts.]

[Bonus! 2 pts.] Identify (V\_1,T\_1)\times\cdots\times(V\_r,T\_r) and V\_{A\_1}\times\cdots\times V\_{A\_r} as k-pairs

Statement

Justification

[1 pt.] Define concatenation map \psi from V\_{A\_1}\times\cdots\times V\_{A\_r} to V\_A

[2 pts.] Show that \psi is an isomorphism of k-vector spaces

Statement

Justification

[1 pt.] State that \psi fits into a commutative square

[3 pts.] Explain why \psi fits into a commutative square

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Commutative square is

\begin{center}

\begin{tikzcd}

k^{n\_1}\times\cdots\times k^{n\_r} \arrow[r, "\psi"] \arrow[d, "T\_{A\_1}\times\cdots\times T\_{A\_r}"'] & k^n \arrow[d, "T\_A"] \\

k^{n\_1}\times\cdots\times k^{n\_r} \arrow[r, "\psi"'] k^n

\end{tikzcd}

\end{center}

Check both compositions and explain why it is sufficient to work with a basis

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Problem 6 [5 pts.]

[1 pt.] State that matrices A,B are k-conjugate iff V\_A \iso V\_B

[1 pt.] Use Problem 5 to show D=diag(a\_1,...,a\_n) satisfies V\_D \iso V\_{[a\_1]}\times\cdots\times V\_{[a\_n]}

[1 pt.] Translate this to an isomorphism V\_D \iso V\_{x-a\_1}\times\cdots\times V\_{x-a\_n}

[2 pts.] Show the equivalence

Problem 7 [5 pts.]

[1 pt.] Reduce isomorphism of pairs to statement about finding suitable basis of k[x]/(q(x))

[1 pt.] Choose appropriate basis {v\_1,...,v\_n}

[1 pt.] Show that mult. by x sends v\_1 to \lambda v\_1

[1 pt.] Show that mult. by x sends v\_j to \lambda v\_j + v\_{j-1} for every j>1

[1 pt.] Show work

Problem 8 = CRT [21 pts.]

Let I\_1,...,I\_n be relatively prime ideals of R

(a)

[1 pt.] Appeal to symmetry to reduce to case r+I\_1=r\_1+I\_1 (or prove other case in similar manner)

[4 pts.] Show that r+I\_1=r\_1+I\_1

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r+I\_1

= r\_2i\_1+r\_1i\_2+I\_1

[1 pt.] = r\_1i\_2+I\_1 since i\_1 \in I\_1 \implies r\_2i\_1 \in I\_1

[1 pt.] = r\_1(1-i\_1)+I\_1 since i\_1+i\_2=1

= r\_1-r\_1i\_1+I\_1

[1 pt.] = r\_1+I\_1 since i\_1 \in I\_1 \implies -r\_1i\_1 \in I\_1

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(b)

[1 pt.] Use fact that I\_k and I\_n are relatively prime to get i\_k in I\_k and i\_{n,k} in I\_n such that i\_k+i\_{n,k}=1

[1 pt.] Write 1 = i\_1...i\_{n-1} + (stuff involving n)

[2 pts.] i\_1...i\_{n-1} is an element of I\_1\cap\cdots\cap I\_{n-1}

Statement

Justification

[2 pts.] (stuff involving n) is an element of I\_n

Statement

Justification

[1 pt.] Conclude that (I\_1\cap\cdots\cap I\_{n-1}) + I\_n = R

(c)

[1 pt.] Base case n=2: content of (a)

Inductive case n>2

[1 pt.] Identify kernel of R\to R\_1\times\cdots R/I\_{n-1} as I\_1\cap\cdots\cap I\_{n-1}

[1 pt.] Translate checking surjectivity of R\to R\_1\times\cdots R/I\_n to checking surjectivity of R\to R/(I\_1\cap\cdots\cap I\_{n-1})\times R/I\_n

[1 pt.] Use (b) to finish the induction

(d)

State that kernel of surjective map in (c) is I\_1\cap\cdots\cap I\_n and use factoring triangle (should have essentially done this to solve (c))

[1 pt.] Show that I\_1...I\_n \subset I\_1\cap\cdots\cap I\_n

Show that I\_1\cap\cdots\cap I\_n \subset I\_1...I\_n

[1 pt.] Base case: trivial since I\_1=I\_1

Inductive case

[1 pt.] Use inductive hypothesis to get I\_1\cap\cdots\cap I\_{n-1} = I\_1...I\_{n-1} = J

Show that J \cap I\_n \subset JI\_n

[1 pt.] Use (b) to conclude that J,I\_n are relatively prime and so find x in J and y in I\_n such that x+y=1

[1 pt.] Write a in J \cap I\_n as ax+ay

[1 pt.] Explain why ax+ay=a is an element of JI\_n

Problem 9 [10 pts.]

Let q\_1(x),...,q\_m(x) in k[x] be pairwise relatively prime monic generating ideals I\_1,...,I\_m

Let q(x) := q\_1(x)...q\_m(x) generating ideal I

Show that I\_i and I\_j are relatively prime for i \neq j

[1 pt.] State the result

[1 pt.] J := I\_i+I\_j is generated by monic r(x) in k[x]

[2 pts.] r(x) | q\_i(x) and r(x) | q\_j(x)

Statement

Justification

[1 pt.] r(x)=1 since q\_i(x) and q\_j(x) are relatively prime (hence J=k[x])

Show that I = I\_1...I\_m

[1 pt.] State the result.

[1 pt.] I \subset I\_1...I\_m since q=q\_1...q\_m

[1 pt.] I\_1...I\_m \subset I since q\_1,...,q\_m are relatively prime

[1 pt.] Use CRT to get isom. of vector spaces

[1 pt.] Explain why we have compatibility of pairs