The Satake Isomorphism and the Langlands Dual

Zachary Gardner

1 Unramified Representations

Given a reductive algebraic group G over a global field F, results like Flath's theorem allow us to glean a lot of important structural information about (irreducible) automorphic representations of $G(\mathbb{A}_F)$ through unramified irreducible representations of $G(F_v)$, where v is a nonarchimedean place of F and a representation (π_v, V_v) of $G(F_v)$ is unramified if $V^{K_v} \neq 0$ for some hyperspecial subgroup $K_v \leq G(F_v)$.

With this in mind, let G be a reductive algebraic group over a nonarchimedean local field F. We assume that G is unramified – i.e., G is quasi-split (has a Borel subgroup) and is split over an unramified finite degree extension of F – and fix a hyperspecial subgroup $K \leq G(F)$. Let (π, V) be an associated unramified irreducible representation of $G(F)$, so that $V^K \neq 0$ and hence V^K generates V in the sense that $\pi(G)V^K = V$. Recall that V^K is naturally a module over the spherical Hecke algebra $C_c^\infty(G(F) /\!\!/ K)$, with associated action

$$\pi(f)v := \int_{G(F)} f(g)\pi(g)v \, dg$$

for dg a Haar measure on $G(F)$.[1] We obtain a map

$$C_c^\infty(G(F) /\!\!/ K) \to \mathrm{End}_{\mathbb{C}}(V^K) \xrightarrow{\sim} \mathbb{C}, \qquad f \mapsto \operatorname{tr} \pi(f)$$

called the **Hecke character** of π.

Remark 1.1. *The term "Hecke character" is often used to refer to an automorphic character of $\mathbb{G}_m = \mathrm{GL}_1$, which is not the same thing as above. The notation $\operatorname{tr} \pi(f)$ suggests that we are taking the trace of a linear operator. Indeed, the $\mathbb{C}$-linear trace of $\pi(f) : V^K \to V^K$ is exactly $\operatorname{tr} \pi(f)$ since $\dim_{\mathbb{C}} V^K = 1$.[2] The latter dimension result is a consequence of Schur's lemma and the commutativity of the spherical Hecke algebra, a fact which follows in general from the Satake isomorphism to be discussed later.*

One of the important features of Hecke characters is captured by the following result.

Theorem 1.2. *Let (π, V) be an unramified irreducible representation of $G(F)$. Then, π is determined up to isomorphism by its Hecke character.*

This theorem is an immediate consequence of the following proposition.

[1] We don't need to specify left or right since $G(F)$ is unimodular.

[2] Later on, we will see an important result of Harish-Chandra on representability of Hecke characters of general admissible irreducible representations.

Proposition 1.3. *Let G be a td group and $K \leq G$ a compact open subgroup. There is an equivalence of categories*

$$\begin{aligned}\{\textit{representations } (\pi, V) \textit{ of } G \textit{ generated by } V^K\} &\longleftrightarrow \{C_c^\infty(G /\!\!/ K)\textit{-modules}\}\\ V &\longmapsto V^K.\end{aligned}$$

Hence, every representation of G generated by V^K is smooth and admissible.

Proof. See Example 6.11 of Conrad's *Smooth representations and Hecke algebras for p-adic groups.* □

An added bonus of this result is that every unramified irreducible representation of $G(F)$ is automatically smooth and admissible.

2 The Satake Isomorphism

Having established the importance of Hecke characters, we now shift our attention to studying the structure of the spherical Hecke algebra $C_c^\infty(G /\!\!/ K)$. The main result of this section will be the construction of the Satake isomorphism, at least in the case that G is split. The next section will focus on the construction of the Langlands dual group ${}^LG$, which allows us to generalize Satake's theorem to the quasi-split case and also formulate Langlands functoriality.

Assume now that G is split (and hence also quasi-split). Let $T \leq G$ be a maximal torus. For ease of computation, we will assume $K = G(\mathcal{O}_F)$ – more general results hold by replacing every instance of $G(\mathcal{O}_F)$ by $G(F) \cap K$ (and doing the same for similar expressions). We have a short exact sequence

$$0 \longrightarrow T(\mathcal{O}_F) \longrightarrow T(F) \xrightarrow{\gamma} X_*(T) \longrightarrow 0$$

of locally compact groups. The map γ is characterized by the condition that $\langle \gamma(t), \chi \rangle = \operatorname{ord}_{\mathfrak{p}} \chi(t)$ for every $t \in T(F)$ and $\chi \in X^*(T)$, where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_F$ and

$$\langle \cdot, \cdot \rangle : X_*(T) \times X^*(T) \to \operatorname{End}(\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Z}, \qquad (\lambda, \chi) \mapsto [\chi \circ \lambda]$$

is the perfect pairing of characters and co-characters.[3] Each choice of uniformizer ϖ for $\mathcal{O}_F$ induces a splitting of this short exact sequence by sending $\lambda \in X_*(T)$ to $\lambda(\varpi) \in T(F)$. For $T = \mathbb{G}_m$ this is just the familiar statement that there is an internal direct product $F^\times = \mathcal{O}_F^\times \times \varpi^{\mathbb{Z}}$. All of this can be made more explicit for $T \cong \mathbb{G}_m^n$ using the isomorphisms

$$\begin{aligned}\mathbb{Z}^n &\xrightarrow{\sim} X_*(T), \qquad (a_1, \dots, a_n) \mapsto \lambda = \lambda_{(a_1,\dots,a_n)} = (t \mapsto (t^{a_1}, \dots, t^{a_n})),\\ \mathbb{Z}^n &\xrightarrow{\sim} X^*(T), \qquad (a_1, \dots, a_n) \mapsto \chi = \chi_{(a_1,\dots,a_n)} = ((t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}).\end{aligned}$$

Given $\lambda = \lambda_{(a_1,\dots,a_n)} \in X_*(T)$ and $\chi = \chi_{(b_1,\dots,b_n)} \in X^*(T)$, we have

$$\langle \gamma(\lambda(\varpi)), \chi \rangle = \operatorname{ord}_{\mathfrak{p}} \chi(\lambda(\varpi)) = \operatorname{ord}_{\mathfrak{p}} \chi(\varpi^{a_1}, \dots, \varpi^{a_n}) = \operatorname{ord}_{\mathfrak{p}} \varpi^{a_1 b_1 + \cdots + a_n b_n} = a_1 b_1 + \cdots + a_n b_n$$

[3] The notation $[\cdot]$ denotes the integer class of an endomorphism, while $\operatorname{ord}_{\mathfrak{p}}$ denotes the nonarchimedean $\mathfrak{p}$-adic valuation on F.

and

$$(\chi \circ \lambda)(s) = \chi(s^{a_1}, \dots, s^{a_n}) = s^{a_1 b_1 + \cdots + a_n b_n} \implies \langle \lambda, \chi \rangle = a_1 b_1 + \cdots + a_n b_n,$$

from which we conclude $\lambda = \gamma(\lambda(\varpi))$. Hence, we have an identification $T(F)/T(\mathcal{O}_F) \cong X_*(T)$. Since $T(F)$ is abelian and $T(F)/T(\mathcal{O}_F)$ is discrete,

$$C_c^\infty(T(F) /\!\!/ T(\mathcal{O}_F)) \cong C_c(T(F)/T(\mathcal{O}_F)) \cong C_c(X_*(T)) \cong \mathbb{C}[X_*(T)],$$

with this identification sending $\lambda \in X_*(T)$ to $\mathbb{1}_{T(\mathcal{O}_F)\lambda(\varpi)T(\mathcal{O}_F)}$ and thus defining an algebra isomorphism. This is the simplest form of the Satake isomorphism.

To handle the spherical Hecke algebra of G and not just T we need to work a little harder. With this in mind, choose a Borel subgroup $B \leq G$ containing T. The choice of B corresponds to a choice of positive roots $\Phi^+ \subseteq \Phi = \Phi(X, T)$, occurring as the representation of $\mathfrak{b} = \mathrm{Lie}(B)$ in the decomposition of the diagonalizable action of T on $\mathfrak{g} = \mathrm{Lie}(G)$. Φ^+ in turn determines a base $\Delta \subseteq \Phi^*$ of simple roots which cannot be written as a sum of two positive roots. We assign to this the data of

$$\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2], \qquad 2\rho = \sum_{\chi \in \Phi^+} \chi \text{ in } X^*(T),$$

and the positive Weyl chamber[4]

$$P^+ := \{\lambda \in X_*(T) : \langle \lambda, \chi \rangle \geq 0 \text{ for every } \chi \in \Phi^+\} = \{\lambda \in X_*(T) : \langle \lambda, \chi \rangle \geq 0 \text{ for every } \chi \in \Delta\}.$$

One advantage of invoking this machinery is the following refinement of the Cartan decomposition.

Theorem 2.1. *We have a decomposition*

$$G(F) = \coprod_{\lambda \in P^+} K\lambda(\varpi)K,$$

where $(\lambda + \mu)(\varpi) = \lambda(\varpi)\mu(\varpi)$.

It follows that the spherical Hecke algebra $C_c^\infty(G(F) /\!\!/ K)$ has a $\mathbb{C}$-vector space basis given by $c_\lambda := \mathbb{1}_{K\lambda(\varpi)K}$ for $\lambda \in P^+$. It is important to note that these functions c_λ do **not** constitute a $\mathbb{C}$-algebra basis of the spherical Hecke algebra. What is true is that

$$c_\lambda * c_\mu = \sum_{\nu \in P^+} d_{\lambda,\mu}(\nu) c_\nu = c_{\lambda+\mu} + \sum_{\nu < \lambda+\mu} d_{\lambda,\mu}(\nu) c_\nu$$

for $d_{\lambda,\mu}(\nu) \in \mathbb{Z}$ and $\leq$ the partial order on P^+ defined by $\lambda < \mu$ if $\mu - \lambda$ is a sum of positive co-roots.[5] The integer $d_{\lambda,\mu}(\nu)$ can be computed explicitly as

$$d_{\lambda,\mu}(\nu) = \#\{(i,j) : \nu(\varpi) \in x_i y_j K\}$$

where $K\lambda(\varpi)K = \coprod_i x_i K$ and $K\mu(\varpi)K = \coprod_j y_j K$.[6] In particular, in the case $G = T$, we have $c_\lambda * c_\mu = c_{\lambda+\mu}$ since double K-cosets in $T(F)$ correspond to single left K-cosets.

[4] This use of the term "Weyl chamber" is dual to the standard usage. The reason for this dual convention will become clear in a little bit. One useful property of P^+ is that it constitutes a complete set of distinct representatives for the Weyl group conjugacy classes of $X_*(T)$.

[5] More precisely, we extend the relation $<$ thus defined to a partial order $\leq$ by forcing reflexivity.

[6] Here we work with left K-cosets because of the choice of Iwasawa decomposition we make below. Decomposing things differently allows us to work with right K-cosets instead.

Let now $N = R_u(B)$ be the unipotent radical of B. We may assume without loss of generality that K, B, T are compatible[7] in the sense that

- $G(F) = B(F)K$;
- $B(F) \cap K = (T(F) \cap K)(N(F) \cap K)$; and
- $T(F) \cap K \leq T(F)$ is maximal compact.

The above Iwasawa decomposition gives $G(F) = T(F)N(F)K$ and we may decompose any choice of Haar measure dg on $G(F)$ via
$$dg = \delta_B(t)\, dtdndk,$$
with
$$dk(K) = 1 = dn(N(F) \cap K)$$
and $\delta_B : B(F) \to \mathbb{R}^{>0}$ the modular quasicharacter characterized by $d(bnb^{-1}) = \delta_B(n)\, dn$, which is trivial on $N(F)$. Phrased another way,
$$\delta_B : B(F) \to \mathbb{R}^{\geq 0}, \qquad b \mapsto |\det{}_{\mathfrak{b}}(b)|_{\mathfrak{p}}\,.$$
Given $f \in C_c^\infty(G(F) /\!\!/ K)$, define $\mathcal{S}f : T(F) \to \mathbb{C}$ by
$$\mathcal{S}f(t) := \delta_B(t)^{1/2} \int_{N(F)} f(tn)\, dn.$$
Getz and Hahn use the notation f^B in place of $\mathcal{S}f$ and call it the **constant term of f along B**. What can we say about $\mathcal{S}f$? It is not too hard to check directly that $\mathcal{S}f$ is compactly supported, locally constant, and left $(T(F) \cap K)$-invariant, hence may be thought of as an element of $C_c^\infty(T(F)/T(F) \cap K) \cong \mathbb{C}[X_*(T)]$. In Lemma 8.6.2, Getz and Hahn prove directly that $\mathcal{S}$ is an algebra homomorphism. The proof is just a computation which is not very enlightening and so we skip it. Note that, given $t = \mu(\varpi) \in T(F)$ for $\mu \in X_*(T)$,
$$\begin{aligned}\delta_B(t)^{1/2} &= |\det(\mathrm{ad}(t) \mid \mathrm{Lie}(N))|_{\mathfrak{p}}^{1/2}\\ &= |2\rho(t)|_{\mathfrak{p}}^{1/2}\\ &= |\varpi^{\langle \mu, 2\rho\rangle}|_{\mathfrak{p}}^{1/2}\\ &= q^{-\langle \mu, \rho\rangle}\end{aligned}$$
for $q := |\mathcal{O}_F/\mathfrak{p}|$. This will help us to understand where $\mathcal{S}$ sends a specific set of nice algebra generators for $C_c^\infty(G(F) /\!\!/ K)$.

Let now $\widehat{G}$ be the complex dual of G, characterized by the fact that the root datum $(X^*(\widehat{T}), X_*(\widehat{T}), \widehat{\Phi}, \widehat{\Phi}^\vee)$ with $\widehat{T}$ the dual torus to T and $\widehat{\Phi} := \Phi(\widehat{G}, \widehat{T})$ is dual to the root datum $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ in the sense that is an isomorphism of root data between $(X^*(\widehat{T}), X_*(\widehat{T}), \widehat{\Phi}, \widehat{\Phi}^\vee)$ and $(X_*(T), X^*(T), \Phi^\vee, \Phi)$. This allows us to view $\mathcal{S}$ as a function from $C_c^\infty(G(F) /\!\!/ K)$ to $\mathbb{C}[X^*(\widehat{T})]$, leading us to the following theorem.

Theorem 2.2 (Satake)**.** *The map $\mathcal{S}$ defines an algebra isomorphism*
$$C_c^\infty(G(F) /\!\!/ K) \cong \mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})(\mathbb{C})}.$$

[7] Getz and Hahn say that K is in **good position** with respect to (B, T).

The content of Satake's theorem is that $\mathcal{S}$ as above is an injective $\mathbb{C}$-algebra homomorphism, with image $\mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G},\widehat{T})(\mathbb{C})}$. In Proposition 8.7.2, Getz and Hahn use the machinery of orbital integrals to prove that $\mathcal{S}$ factors through $\mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G},\widehat{T})(\mathbb{C})}$. We will not address the proof of this here. What we will do is some calculations that suggest the general outline of the proof of Satake's theorem. Suppose we have a decomposition $K\lambda(\varpi)K = \coprod_i x_i K$. Since $G(F) = B(F)K$, we may assume $x_i = t(x_i)n(x_i)$ in $B(F) = T(F)N(F)$. Given $t = \mu(\varpi) \in T(F)$ for $\mu \in X_*(T)$,

$$\begin{aligned}
\mathcal{S}c_\lambda(t) &= \delta_B(t)^{1/2} \int_{N(F)} c_\lambda(tn)\, dn \\
&= q^{-\langle \mu, \rho\rangle} \sum_i dn(N(F) \cap t^{-1}x_i K) \\
&= q^{-\langle \mu, \rho\rangle} \#\{i : t^{-1}t(x_i) \in T(F) \cap K\} \\
&= q^{-\langle \mu, \rho\rangle} \#\{i : t(x_i) \equiv \mu(\varpi) \bmod T(F) \cap K\}.
\end{aligned}$$

In particular, $\mathcal{S}c_\lambda(\lambda(\varpi)) = q^{\langle \lambda, \rho\rangle}$. Moreover, given $\mu \in P^+$,

$$\mathcal{S}c_\lambda(\mu(\varpi)) \neq 0 \implies \mu \leq \lambda.$$

How can we interpret this information? The elements $\lambda \in X_*(T)$, viewed as elements of $X^*(\widehat{T})$, index (isomorphism classes of) highest weight irreducible representations V_λ of $\widehat{G}$. By examining the associated character in $\mathbb{C}[X^*(\widehat{T})]$, which we denote $\operatorname{tr}(V_\lambda)$, we see that the (virtual) representation ring $R(\widehat{G})$ of $\widehat{G}$ is isomorphic as a $\mathbb{C}$-algebra to $\mathbb{C}[X^*(\widehat{T})]$. Using this language, the above computations suggest

$$\mathcal{S}c_\lambda = q^{\langle \lambda, \rho\rangle} \operatorname{tr}(V_\lambda) + \sum_{\mu \in P^+,\ \mu < \lambda} a_\lambda(\mu) \operatorname{tr}(V_\mu)$$

for some coefficients $a_\lambda(\mu) \in \mathbb{C}$ (in fact, in $\mathbb{Z}[q^{\pm 1/2}]$). This expression is invariant under the action of the Weyl group and also demonstrates injectivity of $\mathcal{S}$.

Example 2.3. *If all of this makes your head spin, think of the example $G = \mathrm{GL}_n$. Take T to be the maximal diagonal torus, B the Borel subgroup of invertible upper triangular matrices, and $K = \mathrm{GL}_n(\mathcal{O}_F)$. Given $1 \leq i \leq n$, define $e_i \in X^*(T)$ by $e_i(t_1, \ldots, t_n) := t_i$. We have*

$$\begin{aligned}
\Phi &= \{e_i - e_j : 1 \leq i \neq j \leq n\}, \\
\Phi^+ &= \{e_i - e_j : 1 \leq i < j \leq n\}, \\
\Delta &= \{e_i - e_{i+1} : 1 \leq i < n\}, \\
P^+ &= \{\lambda_{(a_1, \ldots, a_n)} : a_1 \geq \cdots \geq a_n\}.
\end{aligned}$$

The group GL_n is self-dual in the sense that $\widehat{G}$ is GL_n viewed as a $\mathbb{C}$-scheme. We have

$$W(\widehat{G}, \widehat{T})(\mathbb{C}) \cong S_n \implies R(\widehat{G})^{W(\widehat{G},\widehat{T})(\mathbb{C})} \cong \mathbb{C}[z_1, \ldots, z_n]^{S_n} \cong \mathbb{C}[\epsilon_1, \ldots, \epsilon_n],$$

for $\epsilon_1, \ldots, \epsilon_n$ the elementary symmetric polynomials in the n variables $z_1, \ldots, z_n$. Under this isomorphism, ϵ_r is identified with $\wedge^r \underline{\mathbb{C}}^n$, the rth exterior power of the standard representation $\underline{\mathbb{C}}^n$ of GL_n. A $\mathbb{C}$-algebra basis of the spherical Hecke algebra of G is given by c_{λ_r} for $1 \leq r \leq n$ and $c_{\lambda_{-1}}$, where

$$\lambda_r := \lambda_{(1,\ldots,1,0,\ldots,0)}, \qquad \lambda_{-1} := \lambda_{(-1,\ldots,-1)}$$

with r copies of 1 and $n-r$ copies of 0 in the definition of λ_r. One can show that

$$\mathcal{S}c_{\lambda_r} = q^{\langle \lambda_r, \rho \rangle} \operatorname{tr}(\wedge^r \underline{\mathbb{C}}^n) = q^{r(n-r)/2} \operatorname{tr}(\wedge^r \underline{\mathbb{C}}^n).$$

For instance, in the case $n = 2$, we have

$$K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K \sqcup \bigsqcup_{a \in \mathcal{O}_F/\mathfrak{p}} \begin{pmatrix} \varpi & a \\ & 1 \end{pmatrix} K,$$

where the disjoint union runs through a complete set of distinct representatives of $\mathcal{O}_F/\mathfrak{p}$.[8] *It follows that $\mathcal{S}c_{\lambda_1}$ is supported on*

$$\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K_T \sqcup \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K_T$$

for $K_T := T(F) \cap K$ and we have

$$\mathcal{S}c_{\lambda_1} \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} = q^{-\langle \mu_{(0,1)}, \rho \rangle} \cdot 1 = q^{1/2},$$
$$\mathcal{S}c_{\lambda_1} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} = q^{\langle \lambda_1, \rho \rangle} = q^{1/2}.$$

Hence,

$$\mathcal{S}c_{\lambda_1} = q^{1/2} \left(\mathbb{1}_{\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K_T} + \mathbb{1}_{\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K_T} \right) = q^{1/2} \operatorname{tr}(\underline{\mathbb{C}}^2).$$

Similar calculations show $\mathcal{S}c_{\lambda_2} = \operatorname{tr} \wedge^2 \underline{\mathbb{C}}^2$

One very useful consequence of Satake's theorem is the following. We have

$$\mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G},\widehat{T})} \cong \mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})} \cong \mathbb{C}[\widehat{G}]^{\widehat{G}},$$

with the latter isomorphism arising via restriction in accordance with the Chevalley restriction theorem. It follows that $\operatorname{Hom}(\mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})}, \mathbb{C})$ is isomorphic to the set of closed conjugacy classes in $\widehat{G}(\mathbb{C})$ or, equivalently, the set of semisimple conjugacy classes $\widehat{G}^{\mathrm{ss}}(\mathbb{C})/\operatorname{conj}$. Hence, we have a composite isomorphism

$$\operatorname{Hom}(C_c^\infty(G(F) /\!\!/ K), \mathbb{C}) \xrightarrow{(\mathcal{S}^{-1})^*} \operatorname{Hom}(\mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})}, \mathbb{C}) \xrightarrow{\sim} \widehat{G}^{\mathrm{ss}}(\mathbb{C})/\operatorname{conj}$$

identifying irreducible unramified representations of $G(F)$ via their Hecke characters with semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$. For future reference, note that these classes and their eigenvalues go by the name of **Satake parameters**.

3 The Langlands Dual Group

In the previous section, we sketched the proof of Satake's theorem for the case that G is split. To handle the more general case in which G is merely quasi-split, we need to work with a more

[8] For more details, see Proposition 1.4.4 of Bump's *Automorphic Forms and Representations.* Note that Bump works with right K-cosets instead of left K-cosets.

sophisticated object than $\widehat{G}$. This object will be the **Langlands dual group** ${}^L G$, which is also called the **L-group** in connection with the theory of L-functions. In the case that G is split we simply take

$$
{}^L G := \widehat{G}(\mathbb{C}) \times \mathrm{Gal}(F),
$$

for $\mathrm{Gal}(F) := \mathrm{Gal}(F_s/F)$ the absolute Galois group of F and F_s the separable closure of F. In general, ${}^L G$ will be given by a certain semidirect product $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(F)$, with $\mathrm{Gal}(F)$ acting algebraically on $\widehat{G}(\mathbb{C})$ by elements of $\mathrm{Aut}(\widehat{G})$. The presence of $\mathrm{Gal}(F)$ here is not a surprise, but the action of $\mathrm{Gal}(F)$ that we will construct is a little surprising.

With this in mind, let G be a split (connected) reductive algebraic group scheme over an algebraically closed field k. Let $T \leq G$ be a split maximal torus and denote the associated root datum $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ by $\Psi = \Psi(G,T)$. Recall that a choice of Borel subgroup $B \leq G$ containing T defines a set of positive roots $\Phi^+ \subseteq \Phi = \Phi(X,T)$, occurring as the representation of $\mathfrak{b} = \mathrm{Lie}(B)$ in the decomposition of the diagonalizable action of T on $\mathfrak{g} = \mathrm{Lie}(G)$. The set Φ^+ in turn determines a base $\Delta \subseteq \Phi^+$ of simple roots which cannot be written as a sum of two positive roots. Conversely, given a base $\Delta \subseteq \Phi$, each $\alpha \in \Delta$ determines a root group U_α – i.e., a subgroup $U_\alpha \leq G$ uniquely characterized by the fact that

- U_α is normalized by T;
- $U_\alpha \cong \mathbb{G}_a$, the additive group scheme; and
- $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$, the root space.

Example 3.1.

(1) *Let $G = \mathrm{SL}_2$ and T the diagonal torus. The relevant root groups are the strictly upper and lower triangular unipotent subgroups.*

(2) *Let $G = \mathrm{GL}_n$, T the diagonal torus, and $e_i - e_j \in \Phi$. Given a k-scheme R, $U_\alpha(R) = I_n + Re_{ij}$ for e_{ij} the matrix with 1 in the (i,j) position and 0 elsewhere.*

In each of the above examples, T and the relevant root groups generate a Borel subgroup of G. It turns out that this is true in general: the subgroup of G generated by T and U_α for $\alpha \in \Delta$ is Borel. We obtain a bijection between Borel subgroups of G containing T and simple root bases of $\Phi = \Phi(G,T)$. For now, fix a Borel subgroup B containing T – the associated tuple $(\Psi(G,T), \Delta, \Delta^\vee)$ is called a **based root datum** and denoted $\Psi(G,B,T)$. Recall the following classification theorem.

Theorem 3.2 (Chevalley-Demazure)**.** *Let k be an algebraically closed field. Then, every pair (G,T) with G a (connected) reductive algebraic k-group scheme and $T \leq G$ a maximal torus is uniquely determined up to isomorphism by the reduced root datum $\Psi(G,T)$. More specifically, every isomorphism of root data $\Psi(G,T) \cong \Psi(G',T')$ arises from an isomorphism $(G,T) \cong (G',T')$ which is unique up to the conjugation action of $T(k)$ and $T'(k)$. Moreover, every reduced root datum over k is isomorphic to $\Psi(G,T)$ for some pair (G,T) as above.*

Let $\mathrm{Inn}(G)$ denote the inner automorphism group of G, which is isomorphic to $G(k)/\mathcal{Z}G(k)$ via the map sending g to the conjugation automorphism $\mathrm{Ad}(g)$. The classification theorem gives information about the map $\delta : \mathrm{Aut}(G,T) \to \mathrm{Aut}(\Psi(G,T))$ defined as follows. Given $\varphi \in \mathrm{Aut}(G,T)$, $\delta(\varphi)$ acts on the set of $\chi \in X^*(T)$ and $\lambda \in X_*(T)$ via

$$
\delta(\varphi) \cdot \chi := \chi \circ \varphi, \qquad \delta(\varphi) \cdot \lambda := \varphi \circ \lambda.
$$

The classification theorem tells us that δ is surjective, with kernel given by the elements of $\mathrm{Inn}(G)$ arising from $T(k)$. The following proposition is an upgrade of this result.

Proposition 3.3. *There is a short exact sequence* $(*)$ *of groups*

$$1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \xrightarrow{\gamma} \mathrm{Aut}(\Psi(G,B,T)) \longrightarrow 1$$

Proof. Let $\varphi \in \mathrm{Aut}(G)$. The group $\varphi(T)$ is conjugate to T since it is a maximal torus and all maximal tori in G are conjugate. Similarly, $\varphi(B)$ is conjugate to B. Hence, we may choose $g \in G(k)$ so that $\varphi \circ \mathrm{Ad}(g) \in \mathrm{Aut}(G,B,T)$, the choice of g being unique mod $T(k)$. Furthermore, since the action of the Weyl group W on $X^*(T)$ induces a simply transitive action of W on the set of bases of Φ, we may choose $t \in N_G(B)(k) \cap N_G(T)(k) = T(k)$ so that $\varphi \circ \mathrm{Ad}(t) \circ \mathrm{Ad}(g) = \varphi \circ \mathrm{Ad}(tg)$ also preserves Δ. This defines a map $\gamma : \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi(G,B,T))$, which is a well-defined group homomorphism essentially by the properties of the map δ discussed earlier. For a more direct verification, let $\varphi \in \mathrm{Aut}(G)$, $\chi \in X^*(T)$, $\lambda \in X_*(T)$, $t \in T(k)$, and $s \in \mathbb{G}_m(k)$. Suppose that φ acts on $\Psi(G,B,T)$ by $\varphi \circ \mathrm{Ad}(g)$. Then, given any $t_0 \in T(k)$,

$$(\chi \circ \varphi \circ \mathrm{Ad}(t_0 g))(t) = \chi(\varphi(t_0))\chi(\varphi(gtg^{-1}))\chi(\varphi(t_0))^{-1} = \chi(\varphi(gtg^{-1})) = (\chi \circ \varphi \circ \mathrm{Ad}(g))(t)$$

and

$$(\varphi \circ \mathrm{Ad}(t_0 g) \circ \lambda)(s) = \varphi(t_0)\varphi(g\lambda(s)g^{-1})\varphi(t_0)^{-1} = \varphi(g\lambda(s)g^{-1}) = (\varphi \circ \mathrm{Ad}(g) \circ \lambda)(s).$$

Hence, $\varphi \circ \mathrm{Ad}(g)$ and $\varphi \circ \mathrm{Ad}(t_0 g)$ have the same action on $\Psi(G,B,T)$ and so are both valid images for φ under γ. It follows from this that γ is a group homomorphism and we obtain a sequence of the form $(*)$. The classification theorem tells us that γ is surjective. To see that $\ker \gamma = \mathrm{Inn}(G)$, note that clearly $\mathrm{Inn}(G) \subseteq \ker \gamma$. For the reverse containment, let $\varphi \in \ker \gamma$. Then, $\varphi \circ \mathrm{Ad}(tg)$ acts by identity on $\Psi(G,B,T)$ for some $g \in G(k)$ and $t \in T(k)$ and hence by identity on $\Psi(G,T)$. By the classification theorem, $\varphi \circ \mathrm{Ad}(tg) = \mathrm{Ad}(s)$ for some $s \in T(k)$ and so $\varphi = \mathrm{Ad}(tgs^{-1}) \in \mathrm{Inn}(G)$. This proves that $(*)$ is exact. □

Somewhat remarkably, $(*)$ splits as a semidirect product. To see this, we introduce the following notion.

Definition 3.4. *A* ***pinning***, ***framing***, *or* ***épinglage*** *of the triple* (G,B,T) *is a choice of isomorphism* $U_\alpha \cong \mathbb{G}_a$ *for each* $\alpha \in \Delta$. *Equivalently, it is a choice of basis or nonzero vector* $X_\alpha \in \mathfrak{g}_\alpha$ *for each* $\alpha \in \Delta$. *A* ***pinned reductive group*** *is a reductive group equipped with a pinning.*

A choice of pinning should be viewed as a rigidifying constraint, a perspective which will be made more precise in a moment.

Kottwitz, Langlands, and Shelstad refer to pinnings as *splittings*. The following proposition explains this terminology.

Proposition 3.5. *Given pinned reductive groups* $(G,B,T,\{X_\alpha\}_{\alpha\in\Delta})$ *and* $(G',B',T',\{X'_{\alpha'}\}_{\alpha'\in\Delta'})$, *the natural map*

$$\begin{array}{c} \mathrm{Isom}((G,B,T,\{X_\alpha\}_{\alpha\in\Delta}),(G',B',T',\{X'_{\alpha'}\}_{\alpha'\in\Delta'})) \\ \downarrow \\ \mathrm{Isom}(\Psi(G,B,T),\Psi(G',B',T')) \end{array}$$

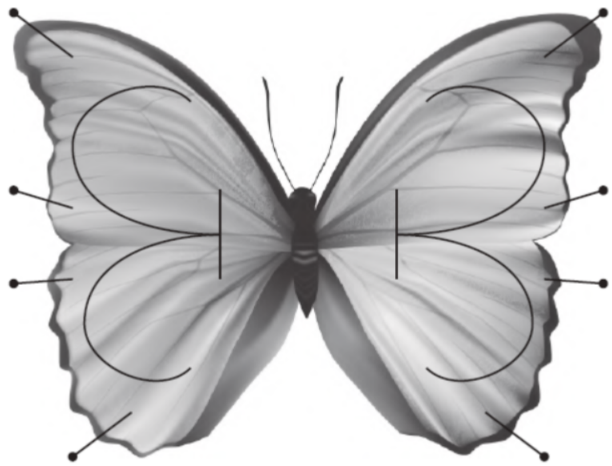

Figure 1: Grothendieck envisioned a pinned reductive group as a butterfly pinned to a board, with the body representing the maximal torus, the wings a pair of opposite Borel subgroups, and the pinning holding everything in place.

is a bijection. Moreover, pinnings of (G, B, T) *are in bijection with (right) splittings of* $(*)$*, up to conjugation by* $T(k)$*.*

Proof. See either Proposition 1.5.5 in Conrad's *Reductive Group Schemes* or Theorem 23.40 in Milne's *Algebraic Groups*. □

Return now to our original setting, in which G is a quasi-split reductive algebraic group over a local (or global) field F. Choose $B \leq G$ Borel containing a maximal torus T. Our goal is to define an action of $\mathrm{Gal}(F)$ on $\mathrm{Aut}(\widehat{G})$, for $(\widehat{G}, \widehat{T})$ complex dual to (G, T). Note first of all that defining an action of $\mathrm{Gal}(F)$ is the same as defining an action of $\mathrm{Aut}(\overline{F}/F)$. This is because the extension $\overline{F}/F_s$ is purely inseparable (assuming it is nontrivial) and so the restriction map $\mathrm{Aut}(\overline{F}/F) \to \mathrm{Gal}(F)$ is necessarily an isomorphism. Put another way, any element $\sigma \in \mathrm{Gal}(F)$ necessarily extends to an automorphism of $\overline{F}$. Each such σ then defines an automorphism of $G_{\overline{F}}$ arising from the automorphism

$$\overline{F} \otimes_F A \to \overline{F} \otimes_F A, \qquad x \otimes a \mapsto \sigma(x) \otimes a$$

for A a finitely generated F-algebra such that $G \cong \mathrm{Spec}(A)$ (recall that G is finite type and necessarily affine). We obtain a group homomorphism $\theta : \mathrm{Gal}(F) \to \mathrm{Aut}(G_{\overline{F}})$. Choose now a Borel subgroup $\widehat{B} \leq \widehat{G}$ containing $\widehat{T}$ corresponding to the dual of the simple base Δ arising from $B_{\overline{F}}$. Consider the composition

$$\mathrm{Gal}(F) \xrightarrow{\theta} \mathrm{Aut}(G_{\overline{F}}) \twoheadrightarrow \mathrm{Aut}(\Psi(G_{\overline{F}}, B_{\overline{F}}, T_{\overline{F}})) \xrightarrow{\sim} \mathrm{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T})) \longrightarrow \mathrm{Aut}(\widehat{G})$$

where the last map arises from a choice of pinning. This defines an action of $\mathrm{Gal}(F)$ on $\widehat{G}(\mathbb{C})$, unique up to conjugation. We then define

$$^{L}G := \widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(F).$$

The composite homomorphism $\mathrm{Gal}(F) \to \mathrm{Aut}(\Psi(G_{\overline{F}}, B_{\overline{F}}, T_{\overline{F}}))$ factors through $\mathrm{Gal}(E/F)$, for E/F the minimal Galois extension for which T_E splits. It follows that the structure of ^{L}G is encoded by $\mathrm{Gal}(E/F)$ and, in the case that G is split, ${}^L G = \widehat{G}(\mathbb{C}) \times \mathrm{Gal}(F)$ in agreement with our original convention.

Example 3.6.

(1) *Let E/F be a finite separable extension and $G := \mathrm{Res}_{E/F}\,\mathrm{GL}_n$, which is the restriction of scalars characterized by*

$$(\mathrm{Res}_{E/F}\,\mathrm{GL}_n)(R) := \mathrm{GL}_n(E \otimes_F R)$$

for R an F-algebra. It turns out that $\widehat{G}(\mathbb{C})$ consists of one copy of $\mathrm{GL}_n(\mathbb{C})$ for each embedding $E \hookrightarrow F_s$ and $\mathrm{Gal}(F)$ acts on $\widehat{G}(\mathbb{C})$ by permutation.

(2) *Let M/F be a quadratic extension and consider the quasi-split unitary group U characterized by*

$$U(R) := \{g \in \mathrm{GL}_n(M \otimes_F R) : J\sigma(g)^{-t}J = g\}$$

for R an F-algebra, J the matrix with 1's on the anti-diagonal, and σ the nontrivial element of $\mathrm{Gal}(M/F)$. It turns out that $\widehat{U} = \mathrm{GL}_n$ and $\mathrm{Gal}(F)$ acts on $\mathrm{GL}_n(\mathbb{C})$ via its quotient $\mathrm{Gal}(M/F)$, which acts via $g \mapsto g^{-t}$.[9]

Remark 3.7. *The above construction did not use any properties of the ground field F, so we obtain a global L-group ${}^L G$ in the case that F is global. For each place v of F there is a local L-group ${}^L G_{F_v}$ related to ${}^L G$ by way of an embedding $\mathrm{Gal}(F_v) \hookrightarrow \mathrm{Gal}(F)$, which is unique up to conjugation.*

Remark 3.8. *So far we have talked briefly about L-groups but said nothing about maps between them. We define a morphism of L-groups to be a homomorphism ${}^L H \to {}^L G$ that is trivial on $\mathrm{Gal}(F)$ and whose associated homomorphism $({}^L H)^0 \to ({}^L G)^0$ of neutral components is induced by a group scheme morphism $\widehat{H} \to \widehat{G}$, $({}^L H)^0$ here just being alternate notation for $H(\mathbb{C})$.*

[9] For more details see Buzzard's short expository notes on unitary groups.