What is $\mathbb{R}$?

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1 Introduction

A number of you have been asking questions about $\mathbb{R}$. What is $\mathbb{R}$? What properties should $\mathbb{R}$ have? Does $\mathbb{R}$ have any equivalent characterizations? Our goal today is to answer these questions and thereby start delving into a big area of mathematics called analysis.

2 The $\mathbb{Q}$ Dilemma

Here at the Ross Math Program we start with an in-depth look at the ring $\mathbb{Z}$. From this knowledge we develop an understanding of the ring $\mathbb{Q}$, constructed as equivalence classes of pairs in $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ under the equivalence relation $(a, b) \sim (c, d) \iff ad = bc$ (which is basically just cross-multiplication). As a countable totally ordered field, $\mathbb{Q}$ has a lot going for it. That being said, $\mathbb{Q}$ has a dilemma. You are probably all familiar with the standard proof that $\sqrt{2}$ is irrational. Though this proof is great, it fails to answer one basic question: Where does $\sqrt{2}$ live? Certainly not in $\mathbb{Q}$, the proof says. The standard refrain is that $\sqrt{2}$ lives in $\mathbb{R}$. And so we come full circle.

Stepping back for a second, why should we expect to encounter a thing like $\sqrt{2}$ when dealing with $\mathbb{Q}$? Let $x \in \mathbb{Q}$ such that $x > 0$ and $x^2 > 2$. Then, a little algebra shows

$$0 < \frac{1}{2}\left(x + \frac{2}{x}\right) < x$$

and

$$\left(\frac{1}{2}\left(x + \frac{2}{x}\right)\right)^2 > 2.$$

Hence, taking $x_0 \in \mathbb{Q}^+$ with $x_0^2 > 2$ and defining

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

for $n \in \mathbb{Z}^{\geq 0}$ gives a strictly decreasing sequence $(x_n)_{n\geq 0}$ all of whose terms are positive with square bigger than 2. If such a sequence had a limit x then x would satisfy $x > 0$ and

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}\left(x + \frac{2}{x}\right). \end{aligned}$$

That is, $x^2 = 2$![1] Unfortunately, this means that the sequence $(x_n)_{n\geq 0}$ cannot converge in $\mathbb{Q}$ and so $\mathbb{Q}$ has a "hole" in it. We will return to this example, which we will affectionately refer to as the "Root Two Example."

Remark 2.1. *If you are not familiar with the notions of convergence and limit in $\mathbb{Q}$, start by observing that $\mathbb{Q}$ has an absolute value $|\cdot|$ which allows us to determine the distance between $x, y \in \mathbb{Q}$ via $|x-y|$. The function $d : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}$ defined by $d(x,y) := |x-y|$ is called a **metric** and satisfies three key properties:*

(i) *Zero Property: $d(x,y) = 0 \iff x = y$;*

(ii) *Symmetry: $d(x,y) = d(y,x)$;*

(iii) *Triangle Inequality: $d(x,z) \leq d(x,y) + d(y,z)$*

*for all $x, y, z \in \mathbb{Q}$. A sequence $(x_n)_{n\geq 0}$ **converges** to or has **limit** $x \in \mathbb{Q}$ if $d(x, x_n)$ can be made arbitrarily small by choosing n large enough. Formally, for every $\epsilon > 0$ there is $N \in \mathbb{Z}^+$ such that $d(x, x_n) < \epsilon$ for every $n \geq N$.[2] A convergent sequence is a sequence that converges.*

3 A First Pass at $\mathbb{R}$

At this point it is good to sit down and think about some of the ingredients that should go into a recipe for $\mathbb{R}$.

- Ring Axioms: $\mathbb{R}$ should be a ring under addition and multiplication.
- Order Axioms: $\mathbb{R}$ should have a notion of ordering and thus positivity.
- Metric: $\mathbb{R}$ should have an absolute value extending the absolute value on $\mathbb{Q}$ in the sense that both absolute values give the same output when evaluated on $\mathbb{Q}$. This should give us a metric d extending the one above.
- Field: Nonzero things in $\mathbb{R}$ should have multiplicative inverses.

You've probably heard that every real number has a decimal expansion. For $0 \leq x < 1$, such an expansion can be thought of as an infinite series

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n},$$

[1] Note that we have implicitly appealed to the continuity of addition, multiplication, and division by nonzero numbers.

[2] We take ϵ to live in $\mathbb{Q}^+$.

where each digit d_n is an element of $\{1, \ldots, 9\}$. We could certainly try to define the real numbers in terms of infinite digit sequences, but we will not pursue this route as it does not generalize well and keeping track of carries gets messy fast. That being said, one key takeaway from the digit sequence picture is that successive truncations provide better and better rational approximations of a given real number.

Example 3.1. *Assuming that we know $\pi \in \mathbb{R}$ with decimal expansion $3.141592\ldots$, we get rational approximations by*

$$(3, 3.1, 3.14, 3.141, \ldots).$$

In fact, we should be able to get rational approximations as good as we want. This is summed up by saying $\mathbb{Q}$ should be **dense** in $\mathbb{R}$. That is, given any $x \in \mathbb{R}$ and $\epsilon > 0$, we should be able find $y \in \mathbb{Q}$ such that $|x - y| = d(x, y) < \epsilon$. Density of $\mathbb{Q}$ will be a consequence of our construction of $\mathbb{R}$.

Now that we have some motivation, we introduce an important definition.

Definition 3.1. *A sequence $(x_n)_{n\geq 0}$ in $\mathbb{Q}$ is called a* ***Cauchy sequence*** *if its terms get arbitrarily close together. Formally, for every $\epsilon > 0$ there is $N \in \mathbb{Z}^+$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$. We denote the set of Cauchy sequences in $\mathbb{Q}$ by $\mathcal{C}(\mathbb{Q})$.*

Exercise 3.1. *Every convergent sequence in $\mathbb{Q}$ is Cauchy. This provides us with a wealth of Cauchy sequences.*

Exercise 3.2. *Define addition on $\mathcal{C}(\mathbb{Q})$ by $(x_n)_{n\geq 0} + (y_n)_{n\geq 0} := (x_n + y_n)_{n\geq 0}$. Define multiplication on $\mathcal{C}(\mathbb{Q})$ similarly. Show that $\mathcal{C}(\mathbb{Q})$ is closed under addition and multiplication.*

Given $x = (x_n)_{n\geq 0} \in \mathcal{C}(\mathbb{Q})$, we have an additive inverse $-x = (-x_n)_{n\geq 0} \in \mathcal{C}(\mathbb{Q})$ and hence a notion of subtraction. Define a relation $\sim$ on $\mathcal{C}(\mathbb{Q})$ by $x \sim y$ if $x - y$ converges with limit 0.

Exercise 3.3. *$\sim$ is an equivalence relation on $\mathcal{C}(\mathbb{Q})$.*

Define $\mathbb{R}$ to be the set of equivalence classes on $\mathcal{C}(\mathbb{Q})$ with respect to $\sim$. We denote the equivalence class of $x \in \mathcal{C}(\mathbb{Q})$ by $[x]$.

Exercise 3.4. *Define addition on $\mathbb{R}$ by $[x] + [y] := [x + y]$. Show that addition is well-defined – that is, given $x \sim x'$ and $y \sim y'$ we have $x + y \sim x' + y'$. Similarly, $[x][y] := [xy]$ gives a well-defined multiplication on $\mathbb{R}$.*

$\mathbb{R}$ contains a copy of $\mathbb{Q}$ consisting of constant sequences (which trivially converge and so are Cauchy). More formally:

Exercise 3.5. *Show that $\mathbb{R}$ is a ring with additive identity and multiplicative identity given by the constant sequences with value 0 and 1, respectively. Define a map $\phi : \mathbb{Q} \to \mathbb{R}$ by $a \mapsto [(a)_{n\geq 0}]$. Show that ϕ is an injective ring homomorphism.*

We can define an absolute value on $\mathbb{R}$ by $|[x]| := [(|x_n|)_{n\geq 0}]$. This is well-defined and extends the absolute value on $\mathbb{Q}$ in the sense that $|\phi(a)| = \phi(|a|)$ for every $a \in \mathbb{Q}$. This gives us a metric on $\mathbb{R}$.

Exercise 3.6. *This exercise delivers on some of the promised properties of $\mathbb{R}$.*

(1) *Show that $\mathbb{Q}$ is dense in $\mathbb{R}$ in the sense that $\phi(\mathbb{Q})$ is dense in $\mathbb{R}$.*

(2) *Show that every nonzero $x \in \mathbb{R}$ has a multiplicative inverse.*

(3) *Define $x \in \mathbb{R}$ to be* ***positive*** *if every $(x_n)_{n\geq 0} \in \mathcal{C}(\mathbb{Q})$ representing x is eventually positive. Show that positivity is a well-defined notion and induces an ordering on $\mathbb{R}$.*

4 WOP for $\mathbb{R}$

We have now given a construction of $\mathbb{R}$ and shown that is a totally ordered field with a dense copy of $\mathbb{Q}$. However, we have not delivered on a promise implied at the beginning – namely, that $\sqrt{2}$ is a positive element of $\mathbb{R}$. We will deliver on this promise using that $\mathbb{R}$ has an analogue of WOP.

Theorem 4.1. *$\mathbb{R}$ is complete – that is, every Cauchy sequence in $\mathbb{R}$ converges.*

Corollary 4.1. *Let $L \subseteq \mathbb{R}$ be a subset which is bounded below by some element of $\mathbb{R}$. Then, L has a* ***greatest lower bound*** *– that is, there exists $\ell \in \mathbb{R}$ such that every lower bound ℓ' of L satisfies $\ell' \leq \ell$. Similarly, a subset $U \subseteq \mathbb{R}$ which is bounded above has a* ***least upper bound*** *– that is, there exists $u \in \mathbb{R}$ such that every upper bound u of U satisfies $u \leq u'$.*

Greatest lower bounds and least upper bounds are necessarily unique if they exist. Using the above notation, we define the **infimum** of L to be $\inf L := \ell$ and the **supremum** of U to be $\sup U := u$.

Exercise 4.1. *Let $(x_n)_{n\geq 0}$ be a sequence in $\mathbb{R}$ which is strictly decreasing and bounded below. Show that $(x_n)_{n\geq 0}$ converges in $\mathbb{R}$.*

Thus, the sequence in the Root Two Example converges in $\mathbb{R}$ and so $\sqrt{2}$ is a positive element of $\mathbb{R}$.

5 Characterizing $\mathbb{R}$

We might try do define $\mathbb{R}$ as the "smallest" (with respect to containment) complete totally ordered field containing a copy of $\mathbb{Q}$. The following exercise shows that this scheme will not work.

Exercise 5.1. *Let $\mathbb{R}(x)$ denote the set of ratios $p(x)/q(x)$ for $p(x), q(x) \in \mathbb{R}[x]$ with $q(x) \neq 0$. Then, $\mathbb{R}(x)$ is a field under the expected operations and, moreover, is a complete totally ordered field with the notion of positivity given by $p/q > 0$ if $a/b > 0$ for $a \in \mathbb{R}$ the leading coefficient of p and $b \in \mathbb{R}$ the leading coefficient of q.*

The following classification theorem tells us something special about $\mathbb{R}$.

Theorem 5.1. *$\mathbb{R}$ is characterized by the property of being an Archimedean complete totally ordered field. That is, if K is an Archimedean complete totally ordered field then there exists an order-preserving field homomorphism $K \to \mathbb{R}$ which is also a homeomorphism (in the sense that it is continuous with continuous inverse).*